Journal of Mathematical Systems, Estimation, and Control
Vol. 4, No. 1, 1994, pp. 3–38
© 1994 Birkhäuser-Boston

Perspective Problems in System Theory and its Application to Machine Vision*

Bijoy K. Ghosh Mrdjan Jankovic Y.T. Wu†

Abstract

This paper introduces observability and identifiability problems that arise in linear dynamical systems with perspective observation function. Such a perspective problem finds its application in the field of Computer Vision specifically in the area of motion estimation of a rigid body with point, line or curve correspondence. The basic result of this paper is to study the correspondence problems in a unifying framework and it is shown that these problems arise as a special case of a more general Perspective System Problem. Problems in perspective system theory introduced in this paper are new.

Key words: perspective, machine vision, motion estimation

1 Introduction

The problem of recovery of three dimensional structure and motion from time varying images have gained increased attention in recent years. Studies on the human visual system have indicated a significant role of motion in the perception process [1], [2]. We, humans, are capable of perceiving the 3-D motion of an object from its 2-D image sequence. Even the perception of 3-D structure is enhanced by the cue of motion. In recent years, Kanade [3], Sugihara [4], Nelson and Young [5] and many others have addressed the problem of 3D shape recovery from a single image.

In this paper we are concerned with the problem of extracting the 3-D motion parameters from a sequence of images. For some of the early research in this area, we refer to [1] - [29]. For a detailed listing of the references see [30, chapter 3] and [31] - [34].

Motivated from the above referenced studies in computer vision, in this paper we introduce a new class of problems in system theory, which we

*Received December 2, 1991; received in final form August 26, 1992.
†Partially supported by DOE under grant No. DE-FG02-90ER14140.

shall call - Perspective Problems in System Theory. The word perspective is derived from vision problems wherein feature points on an object are assumed to be projected perspectively on a screen during the process of imaging. Consequently, the basic problem of perspective system theory that we consider in this paper is to observe the initial condition or to identify the parameters of a dynamical system with the aid of a perspective observation function.

The role of perspective observation has been widely recognized in the field of computer vision. One considers a rigid body which is rotating and translating with unknown but constant velocities. The moving object in turn produces a time varying image on the screen through a system of lenses, possibly a camera. The vision problem considered in this paper is to identify the motion parameters, i.e. the velocities, the initial conditions etc from the moving image.

Typically the motion parameters are identified not from the projection of the entire object but possibly from the projections of various features on the object. In practice however the projection of these features have to be identified on the image and their motions have to be traced. In this paper, the features are considered to be either a point, a straight line or a planar polynomial curve of degree 2. These features are parameterized as points on a suitably defined feature space that has the structure of a manifold. The motion of the rigid body prescribes a dynamical system in the feature space. From the projections of these features on the screen, one constructs an observation function defined on the manifold. In this way we obtain an observed dynamical system defined on a manifold. The problem of computer vision that we consider in this paper is to suitably identify the unknown parameters of the dynamical system and/or observe the initial condition provided that the output function is given for all $t \geq 0$. We now describe a sample problem in details.

Consider a second order state space system given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.1}$$

where we assume that the parameters a_1, a_2, a_3, a_4 are all unknown together with the initial conditions $x_1(0), x_2(0)$. We now consider the output function

$$Y : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{RP}^1 \tag{1.2}$$

defined as

$$(x_1, x_2) \longrightarrow [x_1, x_2]$$

where $[x_1, x_2]$ describes a point in $\mathbb{RP}^1$ in homogeneous coordinates. In other words, the non-zero states (x_1, x_2) of the dynamical system (1.1) is observed up to a homogeneous line, i.e. line through (x_1, x_2) passing

through the origin. $\mathbb{RP}^1$ refers to the space of all homogeneous lines in $\mathbb{R}^2$ (see [35] and [36] for details). We now consider a basic parameter identification problem for perspective systems.

Problem 1.1: Let $(x_1(0), x_2(0)) \neq 0$ and assume that $Y(x_1(t), x_2(t))$ described by (1.2) be given for all $t \geq 0$. Identify the parameters $a_i, i = 1, \ldots, 4$ to the extent possible.

We shall analyze problem 1.1 in full details in section 4 of this paper. For the purpose of this section if we assume $x_2(t) \neq 0$ for all $t \geq 0$ we may write

$$Y(t) = x_1(t)/x_2(t).$$

It is easy to write down the following differential equation treating Y(t) as the independent variable. This is given by

$$\dot{Y} = a_2 + (a_1 - a_4)Y - a_3 Y^2. \tag{1.3}$$

Note that (1.3) is a Riccati equation in one variable. It clearly follows from (1.3) that at best one can hope to identify $a_2, a_1 - a_4$ and a_3. Moreover problem 1.1 is a problem of identifying the parameters of a time invariant Riccati equation. If we define $a = a_2, b = a_1 - a_4, c = -a_3$ we have the following system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} 0 & 1 & Y & Y^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ a \\ b \\ c \end{pmatrix} \tag{1.4}$$

$$Y(t) = (\,1\,0\,0\,0\,)\,(\,x\;a\;b\;c\,)^T. \tag{1.5}$$

In Section 4 of this paper, we shall consider the problem of identifier design for the nonlinear dynamical system (1.4), (1.5). Writing the above system in the form

$$\dot{x}(t) = A(Y)\theta,\ \dot{\theta} = 0,\ Y(t) = x(t), \tag{1.6}$$

the proposed identifier (4.3), we show, identifies the coefficients of a time invariant Riccati Equation. Moreover under a suitable sufficiency condition given by (4.4) or given by Proposition 4.3, the identifier is shown to be globally exponentially convergent.

Unfortunately, not all problems that arises in computer vision can be written in the form (1.6). We now describe the general observation problem that we encounter in all the vision problems that we introduce in this paper.

Consider a linear system

$$\dot{x} = Ax \tag{1.7}$$

where A is a n by n constant matrix, x is a n by 1 state vector given by

$$x = (x_1, x_2, \cdots x_n)^T.$$

Assume $n > 1$ and let m be an integer such that $1 < m < n + 1$. Let us also define

$$B = \{x \in \mathbb{R}^n : \mathrm{x_1} = \mathrm{x_2} = \cdots = \mathrm{x_m} = 0\}.$$

Consider an output function

$$Y : \mathbb{R}^n - \mathrm{B} \to \mathbb{RP}^{m-1} \tag{1.8}$$

defined as

$$(x_1, \ldots, x_n) \longmapsto [x_1, \ldots, x_m].$$

Problem 1.2 (Observation problem): Assume that the matrix A is known. Is it possible to observe $x(0)$ up to a homogeneous line given the output $Y(t)$ for $t \geq 0$?

Problem 1.3 (Parameter identification problem): Assume that the matrix A is unknown and the output $Y(t)$ for $t \geq 0$ is given. The problem is to identify A to the extent possible together with $x(0)$ up to a homogeneous line.

Of course if $n = m = 2$, then the problem 1.3 reduces to problem 1.1. In fact when $n = m$ then problem 1.3 reduces to the problem of identifying the parameters of a Riccati Equation. On the other hand, if $m < n$, the parameter identification problem is more difficult and does not reduce down to observing the parameters of a Riccati Equation.

In Section 3 the problems 1.2 and 1.3 have been satisfactorily analyzed. We obtain a geometric "observability criterion" that completely solves problem 1.2. Furthermore, we show that if the output function $Y(t)$ satisfies a suitable "identifiability condition" then the parameters A can be identified up to the orbits of a suitable group action. The description of the identifiability condition is in general combinatorial. In this paper, we write it down for a generic perspective system. In Section 5, special

cases of such a parameter identification problem have been studied and the identifiability condition has been detailed explicitly.

The main message of this paper is that the point, line and the curve correspondence problems in machine vision reduce to either problems 1.2 or 1.3 under a suitable parameterization of the features. We discuss explicit cases when the vision problem reduces to identifying parameters of a Riccati Equation i.e. when $m = n$. We also consider cases when $m < n$ and discuss explicit methods to identify the motion parameters of a moving rigid body.

2 The Correspondence Problem

We begin this section with an introduction to the point correspondence problem. Assume that a rigid body is undergoing rotation with a constant angular velocity with respect to a fixed axis of rotation. It follows that if (x_t, y_t, z_t) is a point on the rigid body at the time instant t, then

$$(\dot{x_t}\ \dot{y_t}\ \dot{z_t})^T = \Omega\,(x_t\ y_t\ z_t)^T \tag{2.1}$$

where Ω is a skew symmetric matrix given by

$$\begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}. \tag{2.2}$$

Assume that the points on the rigid body are observed perspectively, i.e. every non-zero point is observed up to a line which the point makes with the origin. This is described as

$$Y : \mathbb{R}^3 - \{(0,0,0)\} \rightarrow \mathbb{RP}^2 \tag{2.3}$$

defined as

$$(x_t, y_t, z_t) \longrightarrow [x_t, y_t, z_t].$$

In fact if we assume that the screen is located at $z = 1$, then every point (x, y, z) where $z \neq 0$ is projected on the screen as the point $(x/z, y/z)$. So the projected point indeed identifies the homogeneous line through the point (x, y, z). If we define the screen coordinates to be (x_1, y_1) it follows that $x_1 = x/z, x_2 = y/z$ so that

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \omega_1 x_2 + \omega_2 + \omega_2 x_1^2 + \omega_3 x_1 x_2 \\ -\omega_1 x_1 + \omega_3 + \omega_2 x_1 x_2 + \omega_3 x_2^2 \end{pmatrix}. \tag{2.4}$$

As in (1.3) which is a one dimensional Riccati Equation, the above equation

(2.4) is a two dimensional Riccati Equation. It can be rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y_2 & 1+y_1^2 & y_1 y_2 \\ 0 & 0 & -y_1 & y_1 y_2 & 1+y_2^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

which is of the same form as (1.4), and can be rewritten in the form (1.6)

We now consider a situation where the body is undergoing both rotation and translation. Let us consider two coordinate frames, one attached to the camera (call it (x_c, y_c, z_c)) and the other coordinate frame (call it the body coordinate frame (x_b, y_b, z_b)) with respect to which the body is translating at a constant velocity. Thus we have

$$\begin{pmatrix} \dot{x}_b & \dot{y}_b & \dot{z}_b \end{pmatrix}^T = \begin{pmatrix} \zeta_x & \zeta_y & \zeta_z \end{pmatrix}^T. \tag{2.6}$$

Furthermore assume that

$$\begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T = e^{\Omega t} \begin{bmatrix} x_b & y_b & z_b \end{bmatrix}^T \tag{2.7}$$

where Ω is given by (2.2). Thus we assume that the camera coordinate frame is rotating with respect to the body frame at a constant angular velocity.

With respect to the camera coordinate frame, the dynamics of the body is described as

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = e^{\Omega t} \left[\begin{pmatrix} x_c(0) \\ y_c(0) \\ z_c(0) \end{pmatrix} + \begin{pmatrix} \zeta_x t \\ \zeta_y t \\ \zeta_z t \end{pmatrix} \right]. \tag{2.8}$$

It is not too hard to check that (2.8) can be obtained as a solution to the dynamical system

$$\begin{pmatrix} \dot{X}_c \\ \dot{Z}_c \end{pmatrix} = \begin{pmatrix} \Omega & I \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} X_c \\ Z_c \end{pmatrix} \tag{2.9}$$

where

$$\begin{aligned} X_c(t) &= (x_c(t)\ y_c(t)\ z_c(t))^T \\ Z_c(0) &= (\zeta_x\ \zeta_y\ \zeta_z)^T. \end{aligned}$$

Moreover, if points on the body are observed up to homogeneous lines, we have

$$\begin{aligned} Y &: \mathbb{R}^6 - \mathrm{H} \rightarrow \mathbb{RP}^2 \\ (X_c\, Z_c) &\longmapsto [x_c\, y_c\, z_c] \end{aligned} \tag{2.10}$$

$$H = \{(X_c\ Z_c) :\ X_c = 0\}\,.$$

The pair of equations (2.1), (2.3) and (2.9), (2.10) would be called a perspective system. These are linear systems with perspective observation functions. In particular these are systems of the form (1.7) with observation function (1.8). Note in particular that the dynamical system (2.9) together with the observation function (2.10) corresponds to the problem introduced in section 1 where $n = 6$ and $m = 3$.

Let us now introduce the line correspondence problem. We assume that the moving rigid body has a planar surface and one observes the projection of straight lines on this surface over on a screen. Of course since the rigid body is moving, so does the line on the planar surface of the body and therefore the projection of the line is also moving. We now parameterize the space of straight lines in $\mathbb{R}^3$.

Consider equation of a straight line in the camera co-ordinate frame given by

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.11}$$

where the vectors $(a_1\, a_2\, a_3\, a_4)$ and $(b_1\, b_2\, b_3\, b_4)$ are assumed to be independent. Of course (2.11) defines an affine line in $\mathbb{R}^3$. If we define

$$P = \ row\ span\ of \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \tag{2.12}$$

it follows that P is a two dimensional plane through the origin in $\mathbb{R}^4$. The space of all two dimensional planes through the origin in $\mathbb{R}^4$ is a manifold and is denoted by Grass (2,4) called "the grassmannian manifold of 2 planes in $\mathbb{R}^4$." (See [35] and [36] for details) It may be verified that every point in $Grass(2,4)$ defines an affine line in $\mathbb{R}^3$ and vice versa. Consider a mapping

$$\mathcal{P} \ :\ Grass(2,4) \rightarrow \mathbb{RP}^5 \tag{2.13}$$

defined as follows

$$P \mapsto [\eta_{14}\, \eta_{24}\, \eta_{34}\, \eta_{12}\, \eta_{13}\, \eta_{23}] \tag{2.14}$$

where P is given by (2.12) and η_{ij} is defined as

$$\eta_{ij} = a_i b_j - a_j b_i. \tag{2.15}$$

Note that the coordinates η_{ij} are known as the *Plücker* co-ordinates. Moreover $\mathcal{P}$ is 1-1 but not onto. In fact for a point in $\mathbb{RP}^5$ to have a preimage, the *Plücker* co-ordinates have to satisfy the quadratic condition

$$\eta_{14}\eta_{23} + \eta_{24}\eta_{31} + \eta_{34}\eta_{12} \equiv 0. \tag{2.16}$$

It is easy to argue that as the body moves so does P as a function of time t. In fact

$$P(t) = P(0) \begin{bmatrix} e^{-\Omega t} & \zeta t \\ 0 & 1 \end{bmatrix} \tag{2.17}$$

where $P(t)$ is described by (2.12). Consequently we have

$$\frac{d}{dt}P(t)^T = \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} P(t)^T + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_0 & b_0 \end{bmatrix} \tag{2.18}$$

where

$$\begin{aligned} -a_0 &= \zeta_x a_1(0) + \zeta_y a_2(0) + \zeta_z a_3(0) \\ -b_0 &= \zeta_x b_1(0) + \zeta_y b_2(0) + \zeta_z b_3(0). \end{aligned} \tag{2.19}$$

From (2.18), (2.19), it is fairly straight forward to check that

$$\frac{d}{dt} \begin{bmatrix} \eta_{14} \\ \eta_{24} \\ \eta_{34} \\ \eta_{10} \\ \eta_{20} \\ \eta_{30} \end{bmatrix} = \begin{bmatrix} \Omega & I \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \eta_{14} \\ \eta_{24} \\ \eta_{34} \\ \eta_{10} \\ \eta_{20} \\ \eta_{30} \end{bmatrix} \tag{2.20}$$

$$\eta_{10}(0)\zeta_x + \eta_{20}(0)\zeta_y + \eta_{30}(0)\zeta_z = 0$$

and

$$\begin{pmatrix} \eta_{10}(0) \\ \eta_{20}(0) \\ \eta_{30}(0) \end{pmatrix} = \begin{pmatrix} 0 & -\zeta_z & \zeta_y \\ \zeta_z & 0 & -\zeta_x \\ -\zeta_y & \zeta_x & 0 \end{pmatrix} \begin{pmatrix} \eta_{23}(0) \\ \eta_{31}(0) \\ \eta_{12}(0) \end{pmatrix}$$

where the coordinates η_{ij} are given by (2.15). Thus we have a dynamical system which is, in fact, of the same form as (2.9). We now proceed to describe the observation function associated with the dynamical system (2.20).

If the affine line (2.11) is projected, via the projection operation described by (2.3), on the screen located at $z_c = 1$, one obtains a line given by

$$\eta_{14}x_c + \eta_{24}y_c + \eta_{34} = 0. \tag{2.21}$$

Since we assume that the line (2.21) is observed on the screen, possibly via some edge detection algorithm, it follows that one observes $\eta_{14}, \eta_{24}, \eta_{34}$ up to a non-zero scale factor. Thus the output function is given by

$$Y : \mathbb{R}^6 - \mathrm{H} \to \mathbb{RP}^2 \tag{2.22}$$

$$(\eta_{14}\ \eta_{24}\ \eta_{34}\ \eta_{10}\ \eta_{20}\ \eta_{30}) \mapsto [\eta_{14}\ \eta_{24}\ \eta_{34}]$$

where H is defined in (2.10). Note in particular that if $\eta_{14} = \eta_{24} = \eta_{34} = 0$, the line (2.11) passes through the origin. Such a line is projected on the screen as a point and not a line. It may also be noted that the pair (2.20), (2.22) also describes a perspective system of the same form as (2.9), (2.10).

As a special case, if we assume that the rigid body has no translational velocity, i.e. if $\zeta_x = \zeta_y = \zeta_z = 0$, it follows that $a_0 = b_0 = 0$ and, therefore, $\eta_{10}(t) = \eta_{20}(t) = \eta_{30}(t) \equiv 0$. Thus (2.20) degenerates to the equation

$$\frac{d}{dt}\begin{pmatrix} \eta_{14} \\ \eta_{24} \\ \eta_{34} \end{pmatrix} = \Omega \begin{pmatrix} \eta_{14} \\ \eta_{24} \\ \eta_{34} \end{pmatrix} \tag{2.23}$$

together with the observation function

$$Y : \mathbb{R}^3 - \mathrm{H} \to \mathbb{RP}^2 \tag{2.24}$$

$$(\eta_{14}\ \eta_{24}\ \eta_{34}) \mapsto [\eta_{14}\ \eta_{24}\ \eta_{34}].$$

Thus we obtain a perspective system (2.23), (2.24) which is of the form (2.1), (2.3).

Finally we introduce the curve correspondence problem. Consider a planar second order curve described as

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_6 & b_7 & b_8 & b_9 \end{bmatrix} \begin{bmatrix} x_c^2 \\ y_c^2 \\ z_c^2 \\ x_c y_c \\ x_c z_c \\ z_c y_c \\ x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.25}$$

If (2.25) is projected perspectively on the screen located at $z_c = 1$, we need to parameterize x_c, y_c for which there exists θ such that $\theta x_c, \theta y_c, \theta$ belongs to (2.25). Clearly

$$\begin{gathered} \theta\left(b_6 x_c + b_7 y_c + b_8 z_c\right) = -b_9 \\ \theta^2\left(a_0 x_c^2 + a_1 y_c^2 + a_2 z_c^2 + a_3 x_c y_c + a_4 x_c z_c + a_5 y_c z_c\right) \\ +\theta\left(a_6 x_c + a_7 y_c + a_8 z_c\right) + a_9 = 0. \end{gathered} \tag{2.26}$$

Eliminating θ from (2.26) we have

$$\eta_1 x_c^2 + \eta_2 y_c^2 + \eta_3 x_c y_c + \eta_4 x_c + \eta_5 y_c + \eta_6 = 0$$

where the coordinates η_j can be appropriately computed. Assume that the body undergoes motion of the form (2.8) where we assume furthermore that $\zeta_x = \zeta_y = \zeta_z = 0$. A straightforward calculation yields the following

$$\frac{d}{dt}\begin{bmatrix} \eta_5 \\ \eta_4 \\ \eta_3 \\ \eta_1 \\ \eta_2 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} \Omega & -2R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} \eta_5 \\ \eta_4 \\ \eta_3 \\ \eta_1 \\ \eta_2 \\ \eta_6 \end{bmatrix} \tag{2.27}$$

where Ω is described in (2.2) and R is defined as

$$R = \begin{pmatrix} 0 & -\omega_2 & -\omega_1 \\ -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & 0 \end{pmatrix}.$$

The observation $Y(t)$ is given by

$$Y : \mathbb{R}^6 - (0,0,0,0,0,0) \to \mathbb{RP}^5 \tag{2.28}$$

$$(\eta_1 \cdots \eta_6) \mapsto [\eta_1 \cdots \eta_6].$$

Note once again that the pair (2.27), (2.28) is a perspective system of the form (1.7) where $n = m = 6$.

The planar quadratic curve correspondence problem where $(\zeta_x, \zeta_y, \zeta_y) \neq 0$ has been shown by Wu [37] to be equivalent to observing the parameters of a perspective system of the type (1.7), (1.8) where $n = 21$ and $m = 6$.

To summarize this section, we have introduced various perspective problems that are of interest in machine vision especially in the estimation of motion parameters.

3 The Observability Problem

We have seen in Section 2 that the point, line and planar quadratic curve correspondence problem all fall into a special case of a more general perspective problem considered in (1.7), (1.8). In this section we shall analyze problem 1.2. Let us begin by considering a linear system of the form (1.7) together with an output function of the type

$$Y : \mathbb{R}^n - B \rightarrow \mathbb{RP}^{m-1} \tag{3.1}$$

defined as

$$(x_1, \ldots, x_n) \mapsto [y_1, \ldots, y_m]$$

where we have $y_j = C_j (x_1 \ldots x_n)^T$, $C_1, \ldots, C_m$ being vectors of dimension n and where

$$B = \left\{ (x_1, \ldots, x_n) : C_j (x_1 \ldots x_n)^T = 0, j = 1, \ldots, m \right\}.$$

Let us also assume that $C_1, \ldots, C_m$ are linearly independent. Note that the pair (1.7), (3.1) is a slightly more general class of perspective system. In fact, if

$$C_j = (0\ 0 \ldots 0,\ \underset{\substack{\uparrow \\ j^{th}\ spot}}{1}, 0, \ldots, 0)$$

then the output function (3.1) reduces to (1.8). We now consider the following problem.

Problem 3.1: Find conditions on A, $C_1, \ldots, C_m$ such that it is possible to observe $x(0)$ up to a homogeneous line given the output $Y(t)$ for $t \geq 0$.

Since the vectors $C_i, C_j, i \neq j$ are linearly independent, let us denote by $[C_1, C_j]$ the 2-plane spanned by C_i and C_j. Clearly $[C_i, C_j] \in Grass\,(2, n)$. Moreover $e^{A^T t}\,[C_i\, C_j]$ is a curve in $Grass\,(2, n)$ parameterized by t. We now consider the following definition.

Definition 3.2: A curve $\mathcal{C}_t$ in $Grass(2, n)$ is said to be non-degenerate if there does not exist a $n-2$ dimensional homogeneous plane $\mathcal{W}$ in $\mathbb{R}^n$ such that

$$\mathcal{C}_t \subset \sigma(\mathcal{W})\ \forall t$$

where $\sigma(\mathcal{W})$ is defined as follows

$$\sigma(\mathcal{W}) = \{\mathcal{P} \in Grass(2, n) : \mathcal{P} \cap \mathcal{W}\ is\ nontrivial\}.$$

The main result of this section is described as follows.

Theorem 3.3: *The perspective system (1.7), (3.1) is observable if and only if the set of curves* $e^{A^T t}[C_i, C_j]$, $i \neq j$ $i, j \in \{1, \ldots, m\}$ *are simultaneously non-degenerate, i.e. there does not exist a* $n-2$ *plane* $\mathcal{W}$ *in* $\mathbb{R}^n$ *such that*

$$e^{A^T t}[C_i\, C_j] \subset \sigma(\mathcal{W}) \tag{3.2}$$

for all t *and for all* $i, j \in \{1, 2, \ldots, m\}$, *such that* $i \neq j$.

Before we prove this theorem, we consider the following example.

Example 3.4: Let

$$A = \begin{pmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ \bigcirc & & \lambda_3 \end{pmatrix} \tag{3.3}$$

where we assume that $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Furthermore let

$$\begin{aligned} C_1 &= (\alpha_1\, \alpha_2\, \alpha_3) \\ C_2 &= (\beta_1\, \beta_2\, \beta_3) \end{aligned} \tag{3.4}$$

where we define $\eta_{ij} = \alpha_i\beta_j - \beta_i\alpha_j$. Assume that $\eta_{12}, \eta_{13}, \eta_{23}$ are all non-zero. We now show that the perspective system (1.7), (3.1) with choice of A, C_1, C_2 given by (3.3), (3.4) is observable. Note that in this example we have n=3 and m=2. To see the observability of the perspective system, notice that the curve in Grass(2, 3) referred to in (3.2) is given by column span of

$$\begin{pmatrix} e^{\lambda_1 t}\alpha_1 & e^{\lambda_1 t}\beta_1 \\ e^{\lambda_2 t}\alpha_2 & e^{\lambda_2 t}\beta_2 \\ e^{\lambda_3 t}\alpha_3 & e^{\lambda_3 t}\beta_3 \end{pmatrix}. \tag{3.5}$$

Let W be the vector $(w_1\, w_2\, w_3)^T$. To say that (3.2) is satisfied is to say that

$$det \begin{pmatrix} e^{\lambda_1 t}\alpha_1 & e^{\lambda_1 t}\beta_1 & w_1 \\ e^{\lambda_2 t}\alpha_2 & e^{\lambda_2 t}\beta_2 & w_2 \\ e^{\lambda_3 t}\alpha_3 & e^{\lambda_3 t}\beta_3 & w_3 \end{pmatrix} \equiv 0 \,\forall t \tag{3.6}$$

or equivalently

$$w_1\eta_{23}e^{(\lambda_2+\lambda_3)t} - w_2\eta_{13}e^{(\lambda_1+\lambda_3)t} + w_3\eta_{12}e^{(\lambda_1+\lambda_2)t} \equiv 0.$$

Since $\eta_{12}, \eta_{13}, \eta_{23}$ are nonzero and since $\lambda_1 \neq \lambda_2 \neq \lambda_3$, it follows that $w_1 = w_2 = w_3 = 0$. Thus the curve (3.5) is non-degenerate. The conclusion that the perspective system is observable now follows clearly from Theorem 3.3.

Proof of Theorem 3.3: The perspective system (1.7), (3.1) is not observable if and only if we have a pair of linearly independent initial conditions x_0 and x_0' such that

$$(y_i C_j - y_j C_i)\, e^{At} x_0 = 0 \tag{3.7}$$

and

$$(y_i C_j - y_j C_i)\, e^{At} x_0' = 0 \tag{3.8}$$

for

$$i \neq j,\ i, j \in \{1, \ldots, m\}\,.$$

In (3.7), (3.8) the output vector $[y_1, \ldots, y_m]$ refers to the output of the perspective system for either the input x_0 or x_0'.
Let us define

$$P_t = \textit{column span of } \left(e^{At} x_0, e^{At} x_0'\right).$$

It follows from (3.7) and (3.8) that

$$(y_i C_j - y_j C_i) \in P_t^{\perp} \tag{3.9}$$

for all $t \geq 0$. The above condition (3.9) implies that

$$P_t^{\perp} \textit{ intersects nontrivially the set of planes}$$

$$[C_i\, C_j] \textit{ for each } i \neq j, i, j \in \{1, \ldots, m\}\,. \tag{3.10}$$

Let $\xi_1, \ldots, \xi_{n-2}$ be a set of linearly independent vectors that span $P_0^{\perp}$. It follows that

$$P_t^{\perp} = \textit{column span of } e^{-A^T t}\,[\xi_1, \ldots, \xi_{n-2}] \tag{3.11}$$

where $[\xi_1, \ldots, \xi_{n-2}]$ denotes the $n-2$ plane spanned by $\xi_1, \ldots, \xi_{n-2}$.

From (3.10) and (3.11) it follows that $e^{A^T t}\,[C_i, C_j]$ intersects the plane $[\xi_1, \ldots, \xi_{n-2}]$ non-trivially for each pair $i, j, i \neq j$ and $i, j \in \{1, \ldots, m\}$. If we define $\mathcal{W} = [\xi_1 \ldots \xi_{n-2}]$, the conclusion (3.2) clearly follows.

Conversely if (3.2) is satisfied it follows that $[C_i\, C_j]$ intersects $P_t^{\perp}$ nontrivially for each t. This is because from (3.2) and Definition 3.2, one concludes that $e^{A^T t}\,[C_i, C_j]$ intersects $\mathcal{W}$ nontrivially for every $t \geq 0$. Define independent vectors x_0, x_0' that are in the orthogonal complement of $\mathcal{W}$.

Thus there exists scalers $y_i(t)$, $y_j(t)$ such that $y_j C_i - y_i C_j \in P_t^{\perp}$ for each t. Thus (3.8) is satisfied for each t. Hence

$$\frac{C_i e^{At} x_0}{C_j e^{At} x_0} = \frac{C_i e^{At} x_0'}{C_j e^{At} x_0'}$$

for each t and each $i, j \in \{1, \ldots, m\}$, $i \neq j$. Thus x_0 and x_0' cannot be distinguished by looking at the observation $Y(t)$. (QED)

Remark 3.5: During the time this paper was under review, an interesting observability criterion has been proven for the perspective system (1.7), (3.1). Let us denote the matrix C to be

$$(C_1^T, C_2^T, \ \ldots \ , C_m^T)^T.$$

It has been shown that over the field $\mathbb{C}$ of complex numbers, the perspective system is observable iff

$$\text{rank} \begin{bmatrix} (A-\lambda_i I)(A-\lambda_j I) \\ C \end{bmatrix} = n \tag{3.12}$$

for every pair of eigenvalues λ_i and λ_j (may be the same) of A. Over $\mathbb{R}$ the above condition (3.12) is sufficient for perspective observability. We refer interested readers to [38] for details.

Although in Theorem 3.3 we have obtained a necessary and sufficient condition for the observability of a perspective system, in general checking simultaneous non-degeneracy of the set of curves $e^{A^T t}\,[C_i, C_j]$ may not be trivial. We now address this issue.

Let us construct a set of skew symmetric matrices as follows. Define

$$W_1^{ij} = C_i^T C_j - C_j^T C_i \tag{3.13}$$

and construct

$$W_{k+1}^{ij} = A^T W_k^{ij} - W_k^{ij\,T} A \tag{3.14}$$

for $k = 1, 2, 3, \ldots\ldots$. Note that W_k^{ij} is a set of skew symmetric matrices and therefore can be viewed as a point in $\mathbb{R}^N$, where $N = \left(n^2 - n\right)/2$. We now state and prove the following result.

Theorem 3.6: *A sufficient condition for the perspective system to be observable is that the*

$$\dim\left\{ \ span\left(W_k^{ij}, i \neq j, i, j \in \{1, \ldots, m\}, \ k = 1, 2, \ldots\right)\right\} = N. \tag{3.15}$$

In particular if $n = 3$ then (3.15) is also necessary.

Proof of Theorem 3.6: Consider the perspective system (1.7), (3.1). If the system is not observable, it follows that there exists two linearly independent initial conditions x_0, x_0' such that

$$\frac{C_i e^{At} x_0}{C_j e^{At} x_0} = \frac{C_i e^{At} x_0'}{C_j e^{At} x_0'}. \tag{3.16}$$

The equation (3.16) can be written as

$$x_0^T e^{A^T t} \left[C_i^T C_j - C_j^T C_i \right] e^{At} x_0' = 0. \tag{3.17}$$

By expanding $e^{A^T t}$ and e^{At} in Taylors series and comparing the powers of t one easily concludes that for each i, j if W_k^{ij} is as defined in (3.13), (3.14) we have

$$x_0^T W_k^{ij} x_0' = 0 \tag{3.18}$$

for each $k = 1, 2, \ldots$. $i \neq j, i, j \in \{1, \ldots, m\}$. Let us define

$$x_0 = (\alpha_1, \ldots, \alpha_n)$$
$$x_0' = (\beta_1, \ldots, \beta_n)$$
$$\eta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$$

and rewrite (3.18) as

$$Q\eta = 0$$

where

$$\eta = (\eta_{12}, \ldots, \eta_{1n}, \eta_{23}, \ldots, \eta_{n-1\,n})$$

and where the matrix Q has been appropriately defined. It follows that provided (3.15) is satisfied we have ker Q is empty so that

$$\eta_{ij} = 0 \; i, j \in \{1, \ldots, n\}, \; i \neq j. \tag{3.19}$$

From (3.19) one concludes that x_0 and x_0' are linearly dependent. Thus (3.15) is a sufficient condition for the perspective system to be observable.

Conversely, if (3.15) is not satisfied it follows that there exists a non-zero vector η such that

$$\eta \in Ker\, Q.$$

If $n = 3$ then existence of a non-zero η implies the existence of a corresponding pair of linearly independent vectors x_0, x_0'. Hence the perspective system is not observable. (QED)

Remark 3.7: In general, if $n > 3$, existence of a nonzero $\eta \in Ker\, Q$ does not necessarily imply the existence of linearly independent x_0, x_0' such that the *Plücker* co-ordinates of the 2 plane spanned by x_0, x_0' is given by η. In general, the *Plücker* co-ordinates have to satisfy additional quadratic conditions (similar to (2.16)) in order to have a preimage under *Plücker* embedding (see [39] for details)

Remark 3.8: As a consequence of Theorem 3.6, the following can be said about Example 3.4. If the eigenvalues of A are distinct then a necessary and sufficient condition for the observability of the perspective system is

given by $\eta_{12} \neq 0, \eta_{13} \neq 0$ and $\eta_{23} \neq 0$. To see this, note that (3.19) is satisfied iff the matrix

$$\begin{pmatrix} \eta_{12} & \eta_{13} & \eta_{23} \\ (\lambda_1+\lambda_2)\eta_{12} & (\lambda_1+\lambda_3)\eta_{13} & (\lambda_2+\lambda_3)\eta_{23} \\ (\lambda_1+\lambda_2)^2\eta_{12} & (\lambda_1+\lambda_3)^2\eta_{13} & (\lambda_2+\lambda_3)^2\eta_{23} \end{pmatrix}$$

is non-singular.

As an application of Theorem 3.6, let us now consider the following Example.

Example 3.9: Assume that we have a perspective system where the A matrix is diagonalizable and is given by

$$A = diag\,[\lambda_1\ \lambda_2\ \ldots\ \lambda_{n-1}\ \lambda_n]. \tag{3.20}$$

Let the vectors C_j given by the output function described in (3.1) is given by

$$C_j = (\alpha_{j1}, \ldots, \alpha_{jn})$$

where $j = 1, \ldots, m$. Let us make the following assumptions on the matrices $A,\ C_1, \ldots, C_m$.

$$\begin{array}{c} \textit{Every pair of column vectors in the matrix} \\ [C_1^T, \ldots, C_m^T]^T \textit{ is independent.} \end{array} \tag{3.21}$$

$$\lambda_i + \lambda_j,\ \textit{for } i \neq j,\ i,j \in \{1,2,\ldots,n\} \textit{ are all distinct.} \tag{3.22}$$

We now state and prove the following interesting corollary.

Corollary 3.10: *Consider the perspective system (1.7), (3.1). Assume that the matrix A is diagonalizable and is given by (3.20), after suitable diagonalization. Assume that $C_1, \ldots, C_m$ describes the corresponding output function after diagonalization. Furthermore let λ_j satisfy condition (3.22). A necessary and sufficient condition for observability of the perspective system is given by (3.21).*

Proof of Corollary 3.10: Let us assume that

$$x_0 = (x_{01}, x_{02}, \ldots, x_{on})$$

and

$$x_0' = (x_{01}', x_{02}', \ldots, x_{on}')$$

be a pair of initial conditions such that the corresponding output of the perspective system is the same. Let us now define

$$\begin{aligned} \delta_{ij} &= x_{oi}x'_{oj} - x_{oj}x'_{oi} \\ \theta_{ij}^{k\ell} &= \alpha_{ki}\alpha_{\ell j} - \alpha_{kj}\alpha_{\ell i} \end{aligned}$$

for $i, j \in \{1, \ldots n\}$, $k, \ell \in \{1, \ldots, m\}$. It is a straightforward computation to show that if

$$\frac{C_k e^{At} x_0}{C_\ell e^{At} x_0} = \frac{C_k e^{At} x'_0}{C_\ell e^{At} x'_0}$$

then we have

$$M\Theta^{k\ell}\delta = 0 \tag{3.23}$$

where

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 + \lambda_2 & \lambda_1 + \lambda_3 & \cdots & \lambda_{n-1} + \lambda_n \\ (\lambda_1 + \lambda_2)^2 & (\lambda_1 + \lambda_3)^2 & \cdots & (\lambda_{n-1} + \lambda_n)^2 \\ \vdots & \vdots & \cdots & \vdots \\ (\lambda_1 + \lambda_2)^N & (\lambda_1 + \lambda_3)^N & \vdots\vdots & (\lambda_{n-1} + \lambda_n)^N \end{pmatrix} \tag{3.24}$$

$\Theta^{k\ell} = diag\left(\theta_{12}^{k\ell} \;\; \theta_{13}^{k\ell} \;\; \ldots \;\; \theta_{n-1\,n}^{k\ell}\right)$, $\delta = (\delta_{12} \;\; \delta_{13} \;\; \ldots \;\; \delta_{n-1\;n})^T$ and where $N = n(n-1)/2$. It is well known that under the assumption (3.22), the Vandermonde matrix (3.24) is non-singular. It follows that $\theta_{ij}^{k\ell}\delta_{ij} = 0$ for all $k, \ell \in \{1, \ldots, m\}$, $i, j \in \{1, \ldots, n\}$ and where $k \neq \ell, i \neq j$.

If (3.21) is satisfied, it follows that $\theta_{ij}^{k\ell} \neq 0$ for some k, ℓ and for all i, j. Thus $\delta_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. Hence x_0 and x'_0 are linearly dependent. This implies that the perspective system is observable.

In order to prove the necessary condition we shall apply Theorem 3.3. Note that for the perspective system being considered, the curve referred to in (3.2) is given by

$$\textit{column span of} \begin{bmatrix} \alpha_{k1}e^{\lambda_1 t} & \alpha_{\ell 1}e^{\lambda_1 t} \\ \alpha_{k2}e^{\lambda_2 t} & \alpha_{\ell_2}e^{\lambda_2 t} \\ \vdots & \vdots \\ \alpha_{kn}e^{\lambda_n t} & \alpha_{\ell n}e^{\lambda_n t} \end{bmatrix} \tag{3.25}$$

for $k, \ell \in \{1, \ldots, m\}$. If we assume that (3.21) is not satisfied then it follows that $\theta_{ij}^{k\ell} = 0$ for some i, j and for all k, ℓ. Thus it follows that the set of curves (3.25) are simultaneously degenerate. Thus the perspective system is not observable. (QED)

Example 3.11: Consider the differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.26}$$

together with the output

$$\begin{aligned} Y : \mathbb{R}^2 - \{(0,0)\} &\rightarrow \mathbb{RP}^1 \\ (x_1, x_2) &\mapsto [x_1, x_2]\,. \end{aligned} \tag{3.27}$$

The problem is to identify $x_1(0), x_2(0)$ given the output $Y(t)$ for all $t \geq 0$. Note that apriori the system (3.26) is not in the form of a perspective system introduced in this paper. However (3.26) can be rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.28}$$

where $x_3(0) = 1$. We also consider the output function

$$\begin{aligned} Y : \mathbb{R}^3 - \mathrm{B} &\rightarrow \mathbb{RP}^1 \\ (x_1, x_2, x_3) &\mapsto [x_1, x_2] \end{aligned} \tag{3.29}$$

associated with (3.28) where B is defined as

$$B = \{(x_1, x_2, x_3) : x_1 = x_2 = 0\}\,.$$

If $x_3(0)$ is assumed unknown, then (3.28), (3.29) is a perspective system. A simple calculation applying Theorem 3.6 will show that (3.28), (3.29) is observable if and only if

$$\det \begin{bmatrix} 1 & 0 & 0 \\ a_1 + a_4 & b_2 & -b_1 \\ (a_1 + a_4)^2 & 2a_1b_2 - a_3b_1 + a_4b_2 & a_2b_2 - 2a_4b_1 - a_1b_1 \end{bmatrix} \neq 0. \tag{3.30}$$

Thus a necessary and sufficient condition for observability of the system (3.26) is also given by (3.30) since $x_3(0)$ is known to be 1. It follows that recovering the initial conditions of (3.28) up to a scale factor is equivalent to recovering the initial conditions of (3.26) exactly. Finally note that the condition (3.30) is equivalent to the condition that the pair

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \textit{ is controllable.} \tag{3.31}$$

Thus the affine system (3.26) is observable by the perspective observation function (3.27) if and only if (3.31) is satisfied.

So far in this section, we have studied the problem of observing the initial condition of a perspective system up to a homogeneous line from its output. Such a problem has already been described as problem 1.2 in the introduction. If the parameters $A, C_1, \ldots, C_m$ of the perspective system (1.7), (3.1) are assumed to be apriori unknown, the problem of identifying the parameters of the system together with the initial condition from the output data is called the parameter identification problem. Such a problem has already been introduced as problem 1.3.

We now show that the parameters of a perspective system, under a suitable "identifiability condition" can be identified up to the orbit of a group action. Such a group action is now described.

Consider the space $\mathbb{R}^{n^2+(m+1)n}$ of all triples $(A,\ C_1,\ \ldots, C_m,\ X_0)$ of matrices/vectors that describe perspective systems. We now consider the following group action due to a group $\mathcal{G}$ described as follows

$$\mathcal{G} : \mathbb{R}^{n^2+(m+1)n} \rightarrow \mathbb{R}^{n^2+(m+1)n} \tag{3.32}$$

(a) $(P,\ A,\ C_1, \ldots, C_m,\ X_0) \quad \mapsto \quad (PAP^{-1}, C_1P^{-1}, \ldots, C_mP^{-1}, PX_0)$ where P is a $n \times n$ non-singular matrix.

(b) $(\lambda, A, C_1, \ldots, C_m,\ X_0) \mapsto (A,\ \lambda\, C_1,\ \ldots, \lambda\, C_m,\ X_0)$ where λ is a non-zero real number.

(c) $(\theta,\ A,\ C_1,\ \ldots,\ C_m,\ X_0) \mapsto (\theta I + A,\ C_1, \ldots, C_m,\ X_0)$ where θ is a real number.

It is easy to verify that under the above group action, the output $Y(t)$ of the perspective system remains invariant.

Let us assume furthermore that the pair $(A,\ X_0)$ is controllable i.e., if the column span of $\{X_0,\ AX_0, \ldots, A^{n-1}X_0\}$ has dimension n and the vector $(C_1, C_2, \ldots, C_m)$ is an mn dimensional non-zero vector, then we obtain a canonical form given as follows:

$$A = \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \quad 1 \\ a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & a_2 \quad 0 \end{pmatrix} \tag{3.33}$$

$X_0 = (0\ 0\ \ldots 1)^T$ and $(C_1^*, \ldots, C_m^*)$ is a vector in the direction of $(C_1, \ldots, C_m)$ normalized to have a norm 1.

To see that the above representation is indeed a canonical form, note that if A is an $n \times n$ matrix, one can choose a unique θ such that the characteristic polynomial of $(\theta I + A)$ is of the form

$$s^n + a_2 s^{n-2} + a_3 s^{n-3} + \cdots + a_n .$$

In other words, θ is chosen in such a way that the sum of the real parts of the roots of the characteristic polynomial is zero. Of course if (A, X_0) is a controllable pair, so is $(\theta I + A,\ X_0)$. Thus one can choose a unique non-singular matrix P such that $P(\theta I + A)P^{-1}, PX_0$ is of the above canonical form (3.33). Finally λ can be chosen uniquely up to sign such that the vector $(C_1, \ldots, C_m)$ has norm 1. Note that two different elements of the above canonical form are not in the same orbit of the group $\mathcal{G}$. We, therefore, have the following theorem.

Theorem 3.12: *Consider the space Ω of all triples $(A,\ C_1,\ \ldots, C_m, X_0)$ where the pair (A, X_0) is controllable and the vector $(C_1, \ldots,\ C_m)$ is non-zero. The moduli space $\Omega/\mathcal{G}$ is an analytic manifold of dimension $(m+1)n - 2$ diffeomorphic to $\mathbb{R}^{n-1} \times \mathbb{RP}^{mn-1}$.*

Proof: Consider the map

$$\chi :\ \Omega \to \mathbb{R}^{n-1} \times \mathbb{RP}^{mn-1}$$

given by

$$(A, C_1, \ldots, C_m, X_0) \mapsto (a_2, \ldots, a_n)\,, [C_1, \ldots, C_m]\,.$$

where $[C_1, \ldots, C_m]$ denotes the homogeneous line through the point $(C_1, \ldots, C_m)$ in homogeneous coordinates. It follows that there is a $1-1$ and onto map

$$\bar{\chi} :\ \Omega/\mathcal{G} \to \mathbb{R}^{n-1} \times \mathbb{RP}^{mn-1}$$

obtained as a projection of the map χ on to the moduli space. Finally note that $\bar{\chi}$ is analytic.

(QED)

Example 3.13: Consider the perspective system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.34}$$

together with the output $y(t) = (c_{11}x_1 + c_{12}x_2)\,/\,(c_{21}x_1 + c_{22}x_2)$, where $x_1(0) = 0,\ x_2(0) = 1$. Assume that the vector $(c_{11}\ c_{12}\ c_{21}\ c_{22})$ is non-zero. The perspective system (3.34) is written in a canonical form. It is easy to check that if $a \neq 0$ we have

$$y(t) = \frac{(\sqrt{a}c_{12} + c_{11})\, e^{at} + (\sqrt{a}c_{12} - c_{11})}{(\sqrt{a}c_{22} + c_{21})\, e^{at} + (\sqrt{a}c_{22} - c_{21})} \tag{3.35}$$

and if $a = 0$

$$y(t) = (c_{11}t + c_{12}) / (c_{21}t + c_{22}).$$

An important question to ask is the following. When is it possible to identify the parameters of a perspective system up to orbits of $\mathcal{G}$ action by observing the output $y(t)$? For the system (3.34), it is easy to show that $y(t)$ identifies the parameters of the perspective system up to orbits of the $\mathcal{G}$-action iff $c_{11}c_{22} \neq c_{12}c_{21}$.

Let us now consider the following:

Definition 3.14: The perspective system (1.7), (3.1) is said to be identifiable if no two orbits of the group $\mathcal{G}$ corresponds to the same output $y(t)$ i.e., if $y(t)$ separates points of the moduli space.

The identifiability conditions are now illustrated via the following examples.

Example 3.15: Consider a perspective system with $n = 3$, $m = 2$ described by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} X_0 &= (1\ 1\ 1)^T \\ C_1 &= (1\ c_{12}\ c_{13}) \\ C_2 &= (1\ c_{22}\ c_{23}) \end{aligned} .$$

The output $y(t)$ of the system is given by

$$\frac{1 + c_{12}e^t + c_{13}e^{3t}}{1 + c_{22}e^t + c_{23}e^{3t}} \tag{3.36}$$

which can be expanded in power series as

$$h_0 + h_1 e^t + h_2 e^{2t} + h_3 e^{3t} + \cdots. \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$h_j + c_{22}h_{j-1} + c_{23}h_{j-3} = 0, \qquad j \geq 4.$$

i.e.,

$$(c_{22}\ c_{23}) \begin{pmatrix} h_3 & h_4 \\ h_1 & h_3 \end{pmatrix} = -(h_4\ h_5).$$

If $h_3^2 \neq h_1 h_4$, it follows that c_{22}, c_{23} is unique. Finally note that $h_0 = 1$, $h_1 = c_{12} - c_{22}$, $h_2 = (c_{22} - c_{12})\, c_{22}$, $h_3 = (c_{12} - c_{22})\, c_{22}^2 - c_{23}$, $h_4 = (c_{22} - c_{12})\,(c_{23} + c_{22}^3) + c_{23} c_{22}$. Thus if

$$\left[(c_{12} - c_{22})\, c_{22}^2 - c_{23}\right]^2 \neq (c_{12} - c_{22})\left[(c_{22} - c_{12})\,(c_{23} + c_{22}^3) + c_{23} c_{22}\right] \tag{3.38}$$

then c_{12}, c_{13}, c_{22}, c_{23} are unique. It follows that the exponential rational function (3.36) is a unique representation of the output function. Hence (3.38) is the identifiability condition.

Example 3.16: We now consider a modification of the A matrix in example 3.11 and consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+\pi \end{pmatrix}.$$

The output $y(t)$ of the system is given by

$$\frac{1 + c_{12} e^t + c_{13} e^{\pi t}}{1 + c_{22} e^t + c_{23} e^{\pi T}} \tag{3.39}$$

which can be expanded in a power series as

$$\sum_{i,\, j=0}^{\infty} h_{ij} e^{it} e^{j\pi t}. \tag{3.40}$$

From (3.39) and (3.40) it follows that

$$h_{ij} = -c_{22} h_{i-1\ j} - c_{23} h_{i\ j-1} \tag{3.41}$$

where $i + j \geq 2$. Assume that $h_{ij} = 0$ if either i or j is negative. The equation (3.41) describes a recursion on a 2 dimensional lattice. The equation (3.41) gives rise to the following

$$(c_{22}\ c_{23}) \begin{pmatrix} h_{10} & h_{01} & 0 \\ 0 & h_{10} & h_{01} \end{pmatrix} = -(h_{20}\ h_{11}\ \ h_{02}).$$

For uniqueness of c_{22}, c_{23} it follows that either $h_{10} \neq 0$ or $h_{01} \neq 0$. Thus the perspective system is not identifiable iff $h_{10} = h_{01} = 0$ i.e., iff $c_{12} = c_{22}$ and $c_{13} = c_{23}$ i.e., iff $y(t) \equiv 1$.

Remark 3.17: Identifiability condition for a general perspective system can be written analogously. The details seem to be notationally complicated. However the basic concepts are entirely analogous. Thus the output of a general perspective system can be expanded in a power series which

describes a recursion on a r-dimensional lattice. Here r is the total number of polynomially independent modes of the system. The perspective system is identifiable if the above recursion is unique.

4 Parameter Identification Problem with Full State Observation

In this section, we analyze the parameter identification problem 1.3 for a perspective system of the type (1.7), (1.8). The main result of this section is that if $n = m$, then the problem of identifying parameters of a perspective system is equivalent to the problem of identifying parameters of a Riccati equation. Moreover, there are many identifiers that can be used to identify the parameters of the Riccati equation. Finally, we show that under a suitable sufficiency condition an identifier adopted from Teel [45] is globally convergent.

Let us consider a perspective system of the type (1.7), (1.8). Let us show, first of all, that the proposed parameter identification problem is equivalent to identifying the state vector of a system of the form (1.6). Assume that a_{ij} is the ijth element of the matrix A in (1.7). Assume that $x_n \neq 0$, for all $t \geq 0$. We can therefore write

$$\begin{aligned} \dot{y}_j = \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{jk} y_k \quad & - y_j \left[\sum_{k=1}^{n-1} a_{n_k} y_k \right] + a_{jn} \\ & + (a_{jj} - a_{nn})\, y_j \end{aligned} \tag{4.1}$$

for $j = 1, \ldots, n-1$. Here the co-ordinate y_j is defined as x_j / x_n. Of course, the equation (4.1) is a Riccati equation in $n-1$ variables. Special cases of this equation for $n = 2$ is given by (1.3) and for $n = 3$ is given by (2.4). Let us define the parameter vector θ as the parameters of the Riccati equation given as follows:

$$\theta^T = (\theta_1, \ \ldots, \ \theta_n)$$

$$\theta_j = (a_{j1}, \ldots, a_{j\ j-1}, a_{jj} - a_{nn}, \ a_{j\ j+1}, \ldots, a_{jn})$$

for $j = 1, \ldots n-1$

$$\theta_n = (a_{n1}, \ldots, a_{n\ n-1}) .$$

We can now rewrite the system (4.1) in the compact form given by

$$\dot{y} = A(y)\theta \tag{4.2}$$

where

$$A(y) = \begin{pmatrix} v & & & z_1 \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & v & z_{n-1} \end{pmatrix}$$

$$v = (y_1 \dots y_n \; 1)$$

$$z_j = (y_j y_1 \;\; y_j y_2 \;\; \dots \;\; y_j y_{n-1}) .$$

If we assume that the parameters a_{ij} are constant then we can use an identifier for nonlinear systems to identify these parameters. Few of the applicable identifiers are considered in Teel et al [45]. For example, one can consider an identifier whose realization requires the least number of states. Such an identifier is known in the literature (see [45]) as the observer based identifier. It is given by

$$\begin{aligned} \dot{\hat{y}} &= H(\hat{y} - y) + A(y)\hat{\theta} \\ \dot{\hat{\theta}} &= -A^T P(\hat{y} - y) \end{aligned} \tag{4.3}$$

where H is any Hurwitz matrix (A Hurwitz matrix is a matrix with all its eigenvalues in the open left half of the complex plane.) and where P is a positive definite solution of the Liapunov matrix equation $H^T P + PH = -Q$ and Q is a positive definite symmetric matrix.

Persistence of excitation, a very important concept in the theory of identification and adaptive control, turns out to be necessary and sufficient condition for the convergence of the identification error to zero. This concept is defined below.

Definition 4.1: The matrix $A(y)$ (also called regressor matrix) is said to be persistently exciting (PE) if there exist positive constants α_1, α_2 and σ such that for every t greater than 0 the following inequality is satisfied

$$\alpha_1 I \leq \int_t^{t+\sigma} A^T(y(s))A(y(s))ds \leq \alpha_2 I. \tag{4.4}$$

The above PE condition is sufficient for the convergence of the identifier considered in Teel et al [45] as is evidenced by the following proposition.

Proposition 4.2: *Assume that the matrix $A(y)$ is bounded. Then the parameter estimation error ϕ, of the identifier (4.3) defined via $\phi = \theta - \hat{\theta}$, converges to zero exponentially if and only if the matrix $A(y)$ is persistently exciting.*

For a proof of the above proposition, see section 2.8 of Narendra and Annaswamy [46]. However in general the above PE condition is difficult to check. One useful sufficient condition for the matrix A(y) to be PE is given in Jankovic [44]. In most cases the sufficient condition obtained will be in terms of the initial condition on y and parameter values θ but this is not surprising for the nonlinear system (4.2) because it is well known that the observability (identifiability) of nonlinear systems may hold only for some initial conditions and parameter values. Let us now consider the following matrix

$$\mathcal{P}(t) = \left(A^T(y)\dot{A}^T(y)\ldots A^{(p-1)T}(y)\right)^T. \tag{4.5}$$

In the above formula the notation $A^{(p-1)}$ is used to mean $p-1$ times derivative of A(y). Assume that the rank of $\mathcal{P}^T(t)\mathcal{P}(t)$ is p (equal to the dimension of the parameter vector θ). Then there exist a set of n indices (where n is the dimension of the state vector y) σ_i, $i = 1, \ldots, n$ such that

1) $\sum_{i=1}^{n} \sigma_i = n$ and $0 \leq \sigma_i \leq n$

2) $a_{11}(t), \ldots, a_{1\sigma_1}(t), a_{21}(t), \ldots, a_{n1}(t), \ldots, a_{n\sigma_n}(t)$ are linearly independent at time t

3) $a_{i\sigma_{i+1}} \in \text{span}\{a_{jk} : j = 1, \ldots, n \text{ and } \begin{cases} k = 1, \ldots, \sigma_{i+1} \text{ if } \sigma_j > \sigma_i \\ k = 1, \ldots, \sigma_j \text{ if } \sigma_j \leq \sigma_i \end{cases}\}$

where $a_{ij}(t)$ is the i-th row of $A^{(j-1)}(y(t))$.

The row vectors a_{ij} satisfying the above conditions are called the lexicographic basis and the matrix $\mathcal{P}^*(t)$ defined by

$$\mathcal{P}^*(t) = \left[a_{11}^T(t)\ldots a_{1\sigma_1}^T(t)\ldots a_{n1}^T(t)\ldots a_{n\sigma_n}^T(t)\right]^T$$

is called the lexicographic representation.

Now the following proposition gives the sufficient condition for PE.

Proposition 4.3: *Suppose that the regressor matrix $A(y)$ is bounded together with its first $2p-1$ derivatives. Suppose also that there exist positive constants T, δ and ε such that on every time interval of length T there exists a time instant t^* such that*

1) $|det(\mathcal{P}^T(t^*)\mathcal{P}(t^*))| > 0$

2) a lexicographic representation $\mathcal{P}^*$ exists in a δ neighborhood of each t^* and satisfies

$$|\det\mathcal{P}^*(t^*)| > \varepsilon,$$

then the regressor matrix $A(y)$ is PE.

For a proof of the above proposition, we refer to Jankovic [44].

Remark 4.4: The above Proposition 4.3 holds even when the matrix A is in addition a function of u and t and if it is only piecewise differentiable.

Example 4.5: Let us consider the problem 1.1. For the parameter identification of the Riccati equation (1.3), we propose the use of the parameter identifier given by (4.3). In this example we would like to find conditions that guarantee the convergence of the parameter estimates.

The matrix $\mathcal{P}(t)$ for the system of equation (1.4), (1.5) is given by

$$\mathcal{P}(t) = \begin{pmatrix} 1 & y & y^2 \\ 0 & \dot{y} & 2y\dot{y} \\ 0 & \ddot{y} & 2y\ddot{y} + 2\dot{y}^2 \end{pmatrix}. \tag{4.6}$$

One can calculate $\det\mathcal{P}(t) = 2\dot{y}^3$. If $|\dot{y}| \geq \epsilon > 0$ then it follows by the Proposition 4.3 that the system is PE. It therefore follows that if the system matrix in (1.1)

$$(a_1 - a_4)^2 < -4a_2a_3,$$

then there exists a globally convergent identifier. Note that the above condition implies that the eigenvalues of the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

are complex. By considering other possible combinations of the eigenvalues, it can be shown that the regressor matrix $(1\ y\ y^2)$ is not PE in any other case and therefore the identifiers are not convergent. It therefore turns out that in this example the sufficient condition is also necessary.

Example 4.6: Let us now consider the Riccati equation introduced in (2.4), (2.5), which corresponds to the perspective system (2.1), (2.3). It is easy to see that the phase portraits are periodic. Moreover a sufficiency condition for PE is given by

$$\left| det \begin{pmatrix} y_2 & 1+y_1^2 & y_1y_2 \\ -y_1 & y_1y_2 & 1+y_2^2 \\ \dot{y}_2 & 2y_1\dot{y}_1 & y_1\dot{y}_2 + \dot{y}_1y_2 \end{pmatrix} \right| > \epsilon \tag{4.7}$$

or

$$\left| det \begin{pmatrix} y_2 & 1+y_1^2 & y_1y_2 \\ -y_1 & y_1y_2 & 1+y_2^2 \\ -\dot{y}_1 & \dot{y}_1y_2 + y_1\dot{y}_2 & 2y_2\dot{y}_2 \end{pmatrix} \right| > \epsilon \tag{4.8}$$

sufficiently often in the sense of Proposition 4.3. Since $y_1(t)$ and $y_2(t)$ are periodic functions of time the regressor matrix is not PE only when both determinants are in a sufficiently small neighborhood of zero for all t. Both determinants are 0 if simultaneously

$$\dot{y}_2(1+y_1^2) = \dot{y}_1 y_1 y_2 \tag{4.9}$$
$$\dot{y}_1(1+y_2^2) = \dot{y}_2 y_1 y_2. \tag{4.10}$$

The above two conditions (4.10) are satisfied simultaneously only if $\dot{y}_1 = \dot{y}_2 = 0$, i.e. only if the point does not move on the screen and this will happen only when the point is on the axis of rotation. But in this case the parameters are not identifiable anyway. In order to assure that the two determinants above are bounded away from zero and in view of the fact that the system is periodic we must also exclude a certain small neighborhood of the axis of rotation.

Note in particular that the obtained sufficient condition turns out to be necessary as was the case in Example 4.5.

Remark 4.7: The case when parameters are not constant is more complicated. If we can obtain a model in which parameters change according to a law which is a linear function of parameters and possibly nonlinear function of states of the system, time and the input, we can use the observer proposed in Jankovic [44] to identify these time varying parameters. In other words if the augmented system, (4.2) plus a model for the parameter change, can be written as

$$\dot{y} = A(y)\cdot\theta$$
$$\dot{\theta} = B_1(y)\cdot\theta + B_2(y)$$

then one can construct an observer for the above system given by

$$\dot{\hat{x}} = \bar{A}(y)\cdot\hat{x} + B(y) + K(t)\cdot(y - C\hat{x}) \tag{4.11}$$
$$K(t) = Q(t)C^T R(t) \tag{4.12}$$
$$\dot{Q} = \bar{A}(y)Q + QA^T(y) - QC^T RCQ + L \tag{4.13}$$

where the state vector $x = \begin{bmatrix} y \\ \theta \end{bmatrix}$ and $\hat{x}$ is its estimate, $\bar{A} = \begin{bmatrix} 0 & A(y) \\ 0 & B_1(y) \end{bmatrix}$, $B(y) = \begin{bmatrix} 0 \\ B_2(y) \end{bmatrix}$, $C = [I \quad 0]$, $R(t)$ and $L(t)$ are symmetric uniformly positive definite bounded matrices and the initial condition for the Riccati equation in Q is given by Q_0, a symmetric positive definite matrix.

It can be shown [44] that the above observer is convergent under a suitable condition that is given by a generalization of the PE condition (4.4) to this case. The observer can be applied even in the case when $\bar{A}$

and B are in addition functions of time and input. One nice illustration is obtained if we assume that the perspective projection of a point on a rigid body that rotates freely is available. The rate of change of angular velocities is given by:

$$\begin{aligned} \dot{\omega}_1 &= \alpha\omega_2\omega_3 \\ \dot{\omega}_2 &= \beta\omega_1\omega_3 \\ \dot{\omega}_3 &= \gamma\omega_1\omega_2 \end{aligned} \tag{4.14}$$

where $\alpha = \frac{I_2-I_3}{I_1}$, $\beta = \frac{I_3-I_1}{I_2}$, $\gamma = \frac{I_1-I_2}{I_3}$, I's are moments of inertia.

If one angular velocity is known, then the system (2.4) and (4.14) belongs to the class of systems for which (4.11) is an observer. Moreover it can be shown that the observer is globally convergent for any initial condition provided that ω_1 is the angular velocity assumed known.

5 Parameter Identification Problem with Partial State Observation

In this section we consider once again the parameter identification problem 1.3. So far we have seen in Section 4 that if the entire state of a linear dynamical system is observed up to a homogeneous line, it is possible to state the parameter identification problem as a problem of identifying the parameters of a Riccati Equation from its solution. In particular, the parameters can be identified via an observer which converges exponentially under a suitable sufficiency condition. Unfortunately, if the entire state is not observed up to a homogeneous line, the problem in general does not reduce to observing the parameters of a Riccati equation. We shall see in this section that although the general parameter identification problem is still open, the parameters of the perspective system can be identified in some special cases.

Of course in Section 3 we have already seen that if a suitable "identifiability condition" is satisfied then the parameters of a perspective system (1.7), (3.1) can be identified up to orbits of a group action $\mathcal{G}$. Such a group has already been described in (3.32). We would like to point out, however, that in general it is difficult to check the identifiability condition as prescribed in Section 3 primarily because the condition is not stated geometrically.

The main result of this section is a complete description of the identifiability condition for the perspective system (2.20), (2.22). This is stated as follows.

Theorem 5.1: *Consider the perspective system with state dynamics given by (2.20) and output function given by (2.22). Assume that (2.20) describes*

where $x(0) = x_0 \in \mathbb{R}^3 \setminus \{0\}$, $z(0) = z_0 \in \mathbb{R}^3$ *and where* Ω *is a skew symmetric matrix given by (2.2). Suppose that* $x(t)$ *is observed up to a homogeneous line for all* $t \geq 0$. *From the observation one can estimate* Ω *exactly and* z_0 *up to a homogeneous line iff the vector* ω *is not a linear multiple of* x_0 *or* z_0.

Proof: From (5.2) it follows that

$$x_t = e^{\Omega t}(x_0 + tz_0). \tag{5.3}$$

Since x_t is observed up to a homogeneous line let us realize x_t as

$$x_t = \lambda(t)e^{\Omega' t}(x_0 + tz_0') \tag{5.4}$$

where $\lambda(t)$ is a real valued function and Ω' is a skew symmetric matrix.

(Sufficiency) If $z_0 = 0$ we have

$$x_t = e^{\Omega t}x_0 = \lambda(t)e^{\Omega' t}x_0.$$

From Lemma 5.6 it follows that $\Omega = \Omega'$ provided ω is not a linear multiple of x_0. If $z_0 \neq 0$, it follows from (5.3) and (5.4) that

$$\frac{e^{\Omega t}(x_0 + tz_0)}{\|x_0 + tz_0\|} = \frac{e^{\Omega' t}(x_0 + tz_0')}{\|x_0 + tz_0'\|}. \tag{5.5}$$

In (5.5) if we take the limit as $t \to \infty$ we have for any $\epsilon > 0$, $\exists\, T_\epsilon > 0$ such that for all $t \geq T_\epsilon$

$$\left\| e^{\Omega t}z_0^* - e^{\Omega' t}z_0'^* \right\| < \epsilon \tag{5.6}$$

where $z_0^* = z_0/\|z_0\|$ and $z_0'^* = z_0'/\|z_0'\|$. Since $e^{\Omega t}$ and $e^{\Omega' t}$ are periodic functions of time, it follows that (5.6) must hold for all $t \geq 0$. Since ϵ is arbitrary, we have

$$e^{\Omega t}z_0^* = e^{\Omega' t}z_0'^* \tag{5.7}$$

for all $t \geq 0$. Thus if z_0 is not a linear multiple of ω, then it follows from Lemma 5.6 that $\Omega = \Omega'$ and $z_0^* = z_0'^*$.

Finally if $z_0 \neq 0$ and if z_0 is a linear multiple of ω then from (5.7) it follows that Ω is a linear multiple of Ω'. Moreover, z_0 and z_0' are linear multiples of ω. It follows from (5.3) and (5.4) that

$$e^{(\Omega - \Omega')t}x_0 - \lambda(t)x_0 = t\lambda(t)z_0' - tz_0. \tag{5.8}$$

Since z_0 and z_0' are linearly dependent it follows that the right hand side of (5.8) does not change its direction. Thus we have either $\Omega = \Omega'$ or

the dynamics of a line ℓ, equation of which is given by (2.11). Let P_ℓ denote the plane determined by the line ℓ and the origin. By observing $y(t)$ given by (2.22) it is possible to identify $\omega_1, \omega_2, \omega_3$ exactly and $\zeta_x, \zeta_y, \zeta_z$ up to a plane iff there exist a time t such that the vector $\zeta = (\zeta_x, \zeta_y, \zeta_z)^T \notin P_\ell$ or if the vector $\omega = (\omega_3, -\omega_2, \omega_1)^T$ is not normal to P_ℓ.

Remark 5.2: It may be remarked that if the vectors ζ and ω are not orthogonal to each other then the identifiability condition of Theorem 5.1 is satisfied. In order to prove Theorem 5.1, we shall need to show some basic facts about *Plücker* coordinates. Let ℓ be a line in $\mathbb{R}^3$ described by (2.11). Consider the *Plücker* coordinates described by (2.15). The following facts about *Plücker* coordinates are stated without proof.

Proposition 5.3: *The vector $(\eta_{23}\ \eta_{31}\ \eta_{12})^T$ is in the same direction as that of ℓ (called the direction vector).*

Proposition 5.4: *The vector $(\eta_{14}\ \eta_{24}\ \eta_{34})^T$ is orthogonal to the plane determined by ℓ and the origin. In particular $(\eta_{14}\ \eta_{24}\ \eta_{34})^T$ is orthogonal to $(\eta_{23}\ \eta_{31}\ \eta_{12})^T$.*

Let $\eta_{10}, \eta_{20}, \eta_{30}$ be as defined via (2.15), (2.20). We now state the following.

Proposition 5.5: *The vector $(\eta_{10}(0)\,\eta_{20}(0)\,\eta_{30}(0))^T$ is orthogonal to ζ and $(\eta_{23}(0)\,\eta_{31}(0)\,\eta_{12}(0))^T$ and is equal to the zero vector iff ζ is a linear multiple of $(\eta_{23}(0)\,\eta_{31}(0)\,\eta_{12}(0))^T$.*

Let us now consider the following lemmas.

Lemma 5.6: *Let Ω be a skew symmetric matrix denoted by (2.2) and let n_0 be a non-zero vector in $\mathbb{R}^3$. Suppose that Ω' and n_0' be any other skew symmetric matrix and vector respectively such that*

$$e^{\Omega t} n_0 = k(t) e^{\Omega' t} n_0' \tag{5.1}$$

for all $t \geq 0$ and for some real scalar valued function $k(t)$. Assume that n_0 is not a linear multiple of the vector ω. It follows that $k(t) = k(0) \neq 0$ for all $t \geq 0, n(0) = k(0) n_0'$ and $\Omega' = \Omega$ The proof of Lemma 5.6 is straight forward and is omitted. See Ghosh et al., [42], [43] and Wu [37] for details.

Lemma 5.7: *Consider a dynamical system*

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \Omega & I \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{5.2}$$

x_0 is a linear multiple of ω. But by hypothesis x_0 is not a linear multiple of ω Hence we have $\Omega = \Omega'$.

(Necessity) Suppose that x_0, z_0 and ω are all linear multiples of each other. It follows from (5.3) and (5.4) that

$$x_t = x_0 + tz_0 = \lambda(t)\,(x_0 + tz_0')\,.$$

Consequently, the observation contains no information about Ω and therefore Ω cannot be observed.

(QED)

Let us now consider the proof of Theorem 5.1.

Proof of Theorem 5.1: Let us define $x(t) = (\eta_{14}\,\eta_{24}\,\eta_{34})^T$ and $z(t) = (\zeta_{10}\,\zeta_{20}\,\zeta_{30})^T$. Under this definition of x(t) and z(t), the dynamical system (2.20) reduces to (5.2). From the output function (2.22) it follows that one observers $x(t)$ up to a homogeneous line. Thus it follows from Lemma 5.6 that Ω can be estimated exactly and $(\eta_{10}(0)\,\eta_{20}(0)\,\eta_{30}(0))^T$ up to a homogeneous line iff ω is not a linear multiple of

$$(\eta_{14}(0)\,\eta_{24}(0)\,\eta_{34}(0))^T \tag{5.9}$$

or it is not a linear multiple of

$$(\eta_{10}(0)\eta_{20}(0)\eta_{30}(0))^T\,. \tag{5.10}$$

Note first of all that if $(\eta_{10}(0)\,\eta_{20}(0)\,\eta_{30}(0))^T$ is estimated up to a homogeneous line, it follows from (2.20) that $(\zeta_x\,\zeta_y\,\zeta_z)$ is observed up to a plane which is what has been claimed.

We now have to show that the hypothesis (5.9) and (5.10) is equivalent to the hypothesis of the theorem. Denote

$$d_\ell = (\eta_{23}\,\eta_{31}\,\eta_{12})^T\,.$$

We need to show that $\zeta \in P_\ell$ and $\omega \perp P_\ell \Leftrightarrow \omega, z_0, x_0$ are linear dependent.

($\Rightarrow$) From Proposition 5.4 it follows that $\omega \perp P_\ell \Leftrightarrow$ $x(0)$ is a linear combination of ω.

From Proposition 5.3 we have $d_\ell \in P_\ell$. If d_ℓ and ζ are linearly dependent we conclude from Proposition 5.5 that $z(0) = 0$. Hence z_0 is a linear multiple of ω. If, on the other hand d_ℓ and ζ are linearly independent it follows that span $\{d_\ell, \zeta\}$ is P_ℓ. Using Proposition 5.4 we have z_0 is orthogonal to P_ℓ. Since $\omega \perp P_\ell$ we have z_0 and ω are linear dependent.

($\Leftarrow$) The case when d_ℓ is a linear multiple of ζ is trivial. Assume that d_ℓ and ζ are linear independent. Of course $\omega \perp P_\ell$ follows clearly from Proposition 5.4 and the fact that ω and x_0 are linear dependent. Also since ω and z_0 are linearly dependent, it follows from Proposition 5.5 that $\omega \perp \zeta$. Consequently $\omega \perp$ *span* $\{d_\ell\}$. Thus $P_\ell =$ *span* $\{d_\ell, \zeta\}$.

Hence $\xi \in P_\ell$.

(QED)

In Theorem 5.1 we have obtain necessary and sufficient condition for identification of $(\omega_1, \omega_2, \omega_3)$ exactly and $(\zeta_x, \zeta_y, \zeta_z)$ up to a plane. In the following corollary we show that using a pair of line correspondence one can identify $(\zeta_x, \zeta_y, \zeta_z)$ up to a line.

Corollary 5.8: *Consider the perspective system with state dynamics given by (2.20) and output function given by (2.22). Assume that (2.20) describes the dynamics of a pair of lines ℓ_1 and ℓ_2 each of which satisfy the sufficiency condition of Theorem 5.1. Also assume that the vector $\zeta \neq 0$. Let d_1 and d_2 be respectively the direction vectors of ℓ_1 and ℓ_2 respectively. Then it is possible to identify ω exactly and ζ up to a homogeneous line iff $\zeta \notin$ span (d_1, d_2).*

The proof of Corollary 5.8 is along the same lines as that of Theorem 5.1. The main idea is that if the vector ζ is observed up to a plane using projection of one line one can observe ζ up to a pair of independent planes using projection of two lines. The main idea of the proof is that the two planes are independent. For a proof we refer to the thesis of Wu [37].

6 Conclusion

To conclude, the main results of this paper are now reiterated. This paper introduces the problem of motion estimation with feature correspondence as a problem of identifying the parameters of a dynamical system with perspective observation function. This paper introduces such a perspective systems problem for the first time in the literature. Conditions that guarantee observability and identifiability of such a system have been derived. The main message of this paper is that vision problems by their very nature are perspective. Many geometric structures of a perspective system are lost if they are studied via linearization. Thus one resorts to a nonlinear theory in this paper. A relevant theory for studying motion estimation problems with perspective observation functions is what we call in this paper as Perspective System Theory.

Acknowledgement

The first author would like to acknowledge Prof. Clyde Martin and Prof. W.P. Dayawansa for discussing many problems in system theory that arises in the vision problem during the course of this research. Also acknowledged are the help provided by Dr. P. Allen, Dr. A.K. Bejczy, Dr. R. Bolle, Dr. K. Hashimoto, Dr. H. Kimura and Dr. B. Sridhar for discussing various aspects of the vision problem.

References

[1] J.J. Gibson. *The Perception of the Visual World.* Boston: Houghton Mifflin, 1950.

[2] S. Ullman. *The Interpretation of Visual Motion.* Cambridge, MA: MIT Press, 1979.

[3] T. Kanade. Recovery of the three dimensional shape of an object from a single view, *Artificial Intell.* **17** (1981), 409-460.

[4] K. Sugihara. An algebraic approach to shape from image problems, *Artificial Intell.* **23** (1984), 58-95.

[5] R.N. Nelson and T.Y. Young. Determining three dimensional object shape and orientation from a single perspective view, *Opt. Engineering* **25** (1986), 394-401.

[6] J.W. Roach and J.K. Aggarwal. Determining the movement of objects from a sequence of images, *IEEE Trans. on Pattern Anal. Mach. Intell.* **6** (1980), 554-562.

[7] A. Mitiche and J.K. Aggarwal. Analysis of time varying imagery, *Handbook of Pattern Recognition and Image Processing,* (T.Y. Young and K.S. Fu, eds.) New York: Academic Press, 1986, pp. 311-332.

[8] R.Y. Tsai and T.S. Huang. Estimating three dimensional motion parameters of a rigid planar patch, *IEEE Trans. on ASSP* **29** (1981), 1147-1152.

[9] R.Y. Tsai, T.S. Huang and W.L. Zhu. Estimating three dimensional motion parameters of a rigid planar patch, II: Singular value decomposition, *IEEE Trans. on ASSP* **30** (1982), 525-534; errata, **31** (1983), 514.

[10] R.Y. Tsai and T.S. Huang. Estimating 3-dimensional motion parameters of a rigid planar patch, III: Finite point correspondences and three views problem, *IEEE Trans. on ASSP* **32** (1984), 213-220.

[11] R.Y. Tsai and T.S. Huang. Uniqueness and estimation of three dimensional motion parameters of rigid objects with curved surfaces, *IEEE Transactions on Pattern Analysis and Machine Intelligence,* **6(1)**, January 1984, 13-26.

[12] J.Q. Fang and T.S. Huang. Some experiments on the estimation of motion parameters of a rigid body from two consecutive image frames, *IEEE Trans. on Pattern Anal. Mach. Intell.* **6** (1984), 545-554.

[13] G. Adiv. Determining three dimensional motion and structure from optical flow generated by several objects, *IEEE Trans. on Pattern Anal. Mach. Intell.* **7** (1985), 384-401.

[14] T.Y. Young and Y.L. Wang. Analysis of three dimensional rotation and linear shape changes, *Pattern Recognition Lett.* **2** (1984), 239-242.

[15] K.I. Kanatani. A regional approach to tracking 3-dimensional motion in an image sequence, in *Advances in Computer Vision and Image Processing,* **3** (T.S. Huang, ed.), JAI Press, 1987, pp. 63-99.

[16] K.I. Kanatani. Detecting motion of a planar surface by line and surface integrals, *Computer Vision, Graphics and Image Processing* **29** (1985), 13-22.

[17] W.N. Martin and J.K. Aggarwal. Dynamic scene analysis, *Computer Vision, Graphics and Image Processing* **7** (1978), 356-374.

[18] R. Jain. Extraction of motion information from peripheral processes, *IEEE Trans. Pattern Anal. Mach. Intell.* **3** (1981), 489-503.

[19] A.M. Waxman and S. Ullman. Surface structure and 3-D motion from image flow: kinematic analysis, *Int'l. J. Robotics Research* **4** (1985), 72-94.

[20] S. Amari. Feature spaces which admit and detect invariant signal transformation, in *Proc. 4th Int'l Joint Conf. on Pattern Recognition,* (1978), 452-456.

[21] S. Amari. Invariant structures and feature spaces in pattern recognition problems, *R.A.A.G., Memoirs* **4** (1968), 553-556.

[22] J. K. Aggarwal. Motion and time-varying imagery–An overview, in *Proceedings Workshop on Motion: Representation and Analysis,* May 1986 Charleston, SC.

[23] J. Aloimonos and C.M. Brown. Preception of structure from motion, I: Optical flow v.s. discrete displacements, in *Proc. IEEE Conf. Computer Vision and Pattern Recognition,* June, 1986.

[24] T.J. Broida and R. Chellapa. Performance bounds for estimating three-dimensional motion parameters from a sequence of noisy images, *J. Opt. Soc. Amer.* **A–6**, June 1989.

[25] O.D. Faugeras, F. Lustman, and G. Toscani. Motion and structure from point and line matches, INRIA, BP105, Domain de Volucen, rocquencourt, 78153 Le Chesnay, France, 1987.

[26] R.M. Haralick, H. Joo, C. N. Lee, X. Zhuang, V.G. Vaidya and M.B. Kim. Pose Estimation from corresponding point data, *Machine Vision for Inspection and Measurement.* New York: Academic Press Inc., 1989.

[27] W.C. Hoffman. The lie algebra of visual perspection, *Journal of Mathematical Psychology* **3**, 1966.

[28] B.K.P. Horn. Robot Vision. New York: McGraw-Hill, 1986.

[29] Y. Liu and T.S. Huang. A linear algorithm for motion estimation using straight line correspondences, *Computer Vision, Graphics, and Image Process* **44**, 1988.

[30] S.S. Chen. Image understanding in unstructured environment, series in *Automation* **2**. Riveredge, NJ: World Scientific, 1988.

[31] N. Ayache. *Artificial Vision for Mobile Robots, Stereo Vision and multisensory perception.* Cambridge, MA: The MIT Press, 1991.

[32] Y. Shirai. *Three Dimensional Computer Vision.* New York: Springer Verlag, 1987.

[33] B. K.P. Horn and M.J. Brooks. *Shape from Shading.* Cambridge, MA: The MIT Press, 1989.

[34] R.J. Schalkoff. *Digital Image Processing and Computer Vision.* New York: John Wiley and Sons, 1989.

[35] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* New York: Wiley-Interscience Publication, 1978.

[36] W.M. Boothby. *An Introduction to Differential Manifolds and Riemannian Geometry.* New York: Academic Press, 1975.

[37] Y.T. Wu. *Estimating Three-dimensional Motion Parameters From Feature-based Image Sequence: A Perspective Systems Approach.* Ph.D. Dissertation, Washington University, St. Louis, MO, Dec. 1991.

[38] W. Dayawansa, B. Ghosh, C. Martin and X. Wang. A necessary and sufficient condition for the perspective observability problem, submitted to *Systems and Control Letters.*

[39] J. W. Milnor and J.D. Stasheff. *Characteristic Classes, Annals of Mathematics Studies* **76**. Princeton: Princeton University Press, 1974.

[40] Y. Liu. *Rigid Object Motion Estimation from Intensity Images using Straight Line Correspondences.* Ph.D. Dissertation, University of Illinois at Urbana-Champaign, 1990.

[41] Y. Lin and T.S. Huang. Determination of camera location from 2D to 3D line and point correspondences, *IEEE Transactions on Pattern Analysis and Machine Intelligence* **12(1)**, January 1990.

[42] B. K. Ghosh and Y.T. Wu. Observability of perspective systems: A new approach to computer vision, in *Computation and Control II: Proceedings of the 2nd Bozeman Conference,* Bozeman, Montana, Aug. 1-7, 1990 (K.L. Bowers and J. Lund, eds.).

[43] B.K. Ghosh, Y.T. Wu and M. Jankovic. Some new results in perspective system theory and its application to computer vision, Presented at the ninth Symposium on Energy Engineering Sciences Research on Fluid and Dynamical Systems, 1991.

[44] M. Jankovic. A new observer for a class of nonlinear systems, *Journal of Mathematical Systems, Estimation and Control* **3(2)** (1993), 225-246.

[45] A. Teel, R. Kadiyala, P. Kokotovic and S. Sastry. Indirect techniques for adaptive input-output linearization of non-linear systems, *Int. J. Control* **53(1)** (1991), 193-222.

[46] K.S. Narendra and A.M. Annaswamy. *Stable Adaptive Systems.* Englewood Cliffs, NJ: Prentice Hall, 1989.

DEPARTMENT OF SYSTEMS SCIENCE AND MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MO 63130-4899

Communicated by W.P. Dayawansa