A Realization Theory for Perspective Systems with Applications to Parameter Estimation Problems in Machine Vision

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Abstract— **In this paper we introduce and study a linear dynamical system with a perspective observation function. The study of these systems has been shown to be useful in motion and shape estimation problems in machine vision. We introduce the notion of perspective observability and obtain under a special case a necessary and sufficient condition that would guarantee observability of the initial condition of the linear dynamical system up to a one-parameter magnitude scaling. Subsequently, a new realization theory is introduced which is useful for studying linear systems with perspective observation. Our main result is to show that parameters can be recovered up to orbits of a suitable "perspective group." A new rescaling algorithm is described to identify parameters up to orbits of the perspective system. The identification problem is further analyzed in detail for various problems that are of interest in machine vision.**

I. Introduction to Perspective System Theory

THE CLASS of problems we consider in this paper is motivated from problems in motion and shape estimation well known in computer vision [1], [2]. In recent years, a connection between problems in observability, identifiability, and realization theory, well known in systems theory, has been established with motion and shape estimation problems, well known in machine vision, and these connections have been sketched out in [3] and [4]. In this paper, the interconnection has been further established for a dynamical system that is evolving and is being observed in discrete time.

To introduce a sample problem, consider a linear system with two state variables described as follows:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \tag{1}$$

where we assume that the parameters a_1, a_2, a_3, a_4 are unknown constants. We now consider the output function

$$Y: \mathbb{R}^2 - \{(0,0)\} \to \mathbb{RP}^1 \tag{2}$$

defined as

$$(x_1, x_2) \mapsto [x_1, x_2]$$

where $[x_1, x_2]$ describes a point in $\mathbb{RP}^1$ in homogeneous coordinates, where $\mathbb{RP}^1$ refers to the real projective space of all homogeneous lines in $\mathbb{R}^2$ (see [5] and [6] for details). In other words, we assume that the nonzero state (x_1, x_2) of the dynamical system (1) is observed up to a homogeneous line, i.e., a line through (x_1, x_2) passing through the origin. If we now consider the coordinate chart

$$\{[x_1, x_2]: x_2 \neq 0\}$$

of $\mathbb{RP}^1$ defined by the coordinate

$$y_k = \frac{x_1(k)}{x_2(k)}$$

we obtain the following recursion on y_k given by:

$$y_{k+1} = \frac{a_1 y_k + a_2}{a_3 y_k + a_4}. \tag{3}$$

Note in particular that the right-hand side of (3) is a rational function in y_k. Thus (1) induces a rational recursion in the coordinates y_k of the observed homogeneous lines. We now introduce the following problem.

Problem 1.1 (Sample Identification Problem): Consider the rational dynamical system (3), where we assume that we observe $y_k, k \geq 0$. The problem is to identify parameters a_1, a_2, a_3, a_4 to the extent possible from this data.

It is clear that at best one can hope to identify a_1, a_2, a_3, a_4 up to a one-parameter magnitude scale factor. It would be important to understand, however, under what condition on the parameters a_1, a_2, a_3, a_4 and initial condition $x_1(0), x_2(0)$ would the identification indeed be possible. It may be remarked that Problem 1.1 is motivated classically from the following Riccati question.

Question 1.2 (Riccati's Question): If points on a plane satisfying a linear dynamical system in continuous time are observed up to the slope of the line which the point makes with respect to the origin, what can be said about the dynamics of the slope?

It is well known that in continuous time, the slope dynamics is described by a one-dimensional Riccati equation of the form

$$\dot{\xi} = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 \tag{4}$$

where ξ denotes the slope. Moreover, in discrete time the slope dynamics is given exactly by (3). The rational dynamics (3) can therefore be viewed as a discretization of the Riccati dynamics (4) where the coefficients a_1, a_2, a_3, a_4 have to be appropriately chosen.

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To motivate similar problems from machine vision, we consider a rigid body which is undergoing both rotation and translation. We assume that the body is being observed by a CCD camera which observes points on a surface of the body up to homogeneous lines which the point makes. We consider two coordinate frames, one attached to the camera [let us call it (x_c, y_c, z_c)] and the other to the coordinate frame [let us call it the body coordinate frame (x_b, y_b, z_b)] with respect to which the body is translating at a constant velocity. Thus we have

$$\begin{pmatrix} x_b(k+1) \\ y_b(k+1) \\ z_b(k+1) \end{pmatrix} = \begin{pmatrix} x_b(k) \\ y_b(k) \\ z_b(k) \end{pmatrix} + \begin{pmatrix} \zeta_x \\ \zeta_y \\ \zeta_z \end{pmatrix}. \tag{5}$$

Furthermore, let us assume that the camera coordinate frame is rotating with respect to the body frame at a constant angular velocity. Thus we have

$$\begin{pmatrix} x_c(k) \\ y_c(k) \\ z_c(k) \end{pmatrix} = e^{\Omega k} \begin{pmatrix} x_b(k) \\ y_b(k) \\ z_b(k) \end{pmatrix} \tag{6}$$

where Ω is a skew symmetric matrix given by

$$\Omega \triangleq \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0. \end{pmatrix}. \tag{7}$$

Combining (5) and (6) we obtain

$$\begin{pmatrix} \mathcal{X}_c(k+1) \\ \mathcal{Z}_c(k+1) \end{pmatrix} = \begin{pmatrix} e^{\Omega} & e^{\Omega} \\ 0 & e^{\Omega} \end{pmatrix} \begin{pmatrix} \mathcal{X}_c(k) \\ \mathcal{Z}_c(k) \end{pmatrix} \tag{8}$$

where $\mathcal{X}_c = (x_c \quad y_c \quad z_c)^T$ and where $\mathcal{Z}_c(0) = (\zeta_x \quad \zeta_y \quad \zeta_z)^T$. Equation (8) describes the combined effect of rotation and translation of the rigid body. This example has already been considered in [3] for systems described in continuous time.

Since the CCD camera observes the point $(x_c \quad y_c \quad z_c)$ up to a homogeneous line, we have the following observation function:

$$\begin{aligned} Y: \mathbb{R}^6 - H &\rightarrow \mathbb{RP}^2 \\ (\mathcal{X}_c \quad \mathcal{Z}_c) &\mapsto [x_c \quad y_c \quad z_c] \end{aligned} \tag{9}$$

where

$$H \triangleq \{(\mathcal{X}_c \quad \mathcal{Z}_c) : \mathcal{X}_c = 0\}.$$

The pairs (1), (2) and (8), (9) describe a linear system with perspective observation and can be considered as examples of perspective dynamical systems studied in this paper. In general, let us consider a linear system

$$x_{k+1} = Ax_k, \quad z_k = Cx_k \tag{10}$$

where $x_k \in \mathbb{R}^n, z_k \in \mathbb{R}^p$ for $k = 0, 1, \cdots$. Let us assume that $p > 1$ and that z_k is observed only up to a homogeneous line, i.e., we have the observation function

$$\begin{aligned} Y: \mathbb{R}^p - \{0\} &\rightarrow \mathbb{RP}^{p-1} \\ z_k &\mapsto [z_k] = [Cx_k] \end{aligned} \tag{11}$$

where $[z_k]$ is a point in the projective space $\mathbb{RP}^{p-1}$ of all homogeneous lines in $\mathbb{R}^p$. The observation function Y is not defined for $z_k = 0$. In this paper we shall allow z_k to be the zero vector at certain time instants $k = k_1, k_2, \cdots$. Furthermore, we shall not distinguish between two dynamical systems that have the same value of the observation function Y except perhaps the observation function is not defined at $k = k_1, k_2, \cdots$. The pair (10), (11) would be referred to as a "perspective dynamical system." We now propose to introduce problems 1.3 and 1.6 as follows.

Problem 1.3 (Perspective Observability Problem): The dynamical system (10), (11) is said to be perspectively observable if for every pair of initial conditions x_0, x_0' where $[x_0] \neq [x_0']$, there exists some k such that $[CA^k x_0]$ and $[CA^k x_0']$ are defined and $[CA^k x_0] \neq [CA^k x_0']$. We seek condition on A, C such that the dynamical system (10), (11) is perspectively observable.

In Section II we show the following result.

Theorem 1.4: The perspective system (10), (11) is perspectively observable if for any set of eigenvalues λ_0, λ_1 of A (possibly repeated) one has

$$\operatorname{rank} \begin{pmatrix} (A - \lambda_0 I)(A - \lambda_1 I) \\ C \end{pmatrix} = n. \tag{12}$$

Moreover, if the eigenvalues of A are in $\mathbb{R}$, (12) is also necessary.

Remark 1.5: Theorem 1.4 is true even over the base field $\mathbb{C}$ of complex numbers. In fact, since every eigenvalue of A is in $\mathbb{C}$, (12) would always be necessary and sufficient for perspective observability over $\mathbb{C}$.

It may be noted that a result similar to Theorem 1.4 has already been introduced for perspective systems described in continuous time [7], [8]. As a corollary of Theorem 1.4 we infer that if the perspective system (8), (9) is perspectively observable, then provided that the rotation matrix Ω is known, one can recover the vector

$$(x_c(0) \quad y_c(0) \quad z_c(0) \quad \zeta_x \quad \zeta_y \quad \zeta_z)$$

up to a one-parameter magnitude scaling. Of course if Ω is unknown, one introduces the following identification problem.

Problem 1.6 (Perspective Identification Problem): Consider the dynamical system (10), (11). Assume that $x_0 \neq 0$ and the parameters A, C are unknown. For some k_1, k_2, where $0 \leq k_1 < k_2$, assume that $[CA^k x_0], k = k_1, k_1 + 1, \cdots, k_2$ has been observed. The problem is to compute A, C, and $[x_0]$ from this data to the extent possible.

It may be noted that the parameters $A, C, [x_0]$ cannot be completely recovered from $[z_k] = [CA^k x_0]$. The main result of this paper is to show that the nonuniqueness in $C, A, [x_0]$ which produces the same $[z_k]$ can be described as an orbit of a perspective group $\mathcal{G}$. The exact description of $\mathcal{G}$ not only depends on the McMillan degree n of system (10), (11) but additionally on two other invariants g and g_1. Definitions of these invariants are relegated to Section III of this paper.

For the purpose of this section, we state that for a generic choice of the matrix A, specifically when the characteristic polynomial of A have all nonzero coefficients, the invariants

g and g_1 take the value 1 and 0. In this case, the orbit of the perspective group $\mathcal{G}$ is described as follows.

1) $P \in GL(n)$ acting on (C, A, x_0) as follows:

$$(C, A, x_0) \mapsto (CP, P^{-1}AP, P^{-1}x_0). \tag{13}$$

2) $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} - \{0\}$ acting on (C, A, x_0) as follows:

$$(C, A, x_0) \mapsto (\lambda_1 C \quad \lambda_2 A \quad \lambda_3 x_0). \tag{14}$$

Action (13) is already well known in linear system theory and is obtained as a result of change of basis in the state space. Scaling action (14) is new and is a result of the perspective observation function (11).

The collective actions (13), (14) will be referred to as the action due to a perspective group $\mathcal{G}$. In Section III we show the following important result for the case when the number of outputs p is greater than or equal to the number of state variables n.

Theorem 1.7: Let us consider triplets (C, A, x_0) such that the vectors $Cx_0, CAx_0, \cdots, CA^{n-1}x_0$ are linearly independent and such that the pair of integers g, g_1 as defined in Section III are, respectively, 1, 0. Under this generic condition on the triplet (C, A, x_0), it is possible to identify the parameters of system (10), (11) up to orbits of the perspective group $\mathcal{G}$.

In fact, for a continuous time system, a result similar to Theorem 1.7 has already been introduced and proven in [3]; see also [9].

Remark 1.8: Let us define two dynamical systems of the form (10) to be perspectively equivalent if their outputs z_k differ by a nonzero multiplicative factor for each $k = 0, 1, \cdots$. Theorem 1.7 characterizes all dynamical systems of minimal state dimension n that are perspectively equivalent to a given one under the condition that $Cx_0, CAx_0, \cdots, CA^{n-1}x_0$ are linearly independent vectors, and where the integers $g = 1$ and $g_1 = 0$. In this paper, we shall also extend the result stated in Theorem 1.7 to the general case when $g \geq 1$ and $g_1 \geq 0$. This has been described in Theorem 3.3.

It has been observed in Section III that the description of the group $\mathcal{G}$ and its action on the perspective system (10), (11) is somewhat straightforward if one were to describe the group action on the sequence $Cx_0, CAx_0, CA^2x_0, \cdots$ as opposed to the triplet (C, A, x_0). This leads to an important question as to how one would realize in state space a sequence of nonzero vectors as homogeneous coordinates of the output of a perspective system. In particular, given a realization of a perspective system, is it possible to parameterize the realization of the orbit of the perspective group acting on the system. This way one would parameterize all systems preserving a certain triplet of integer invariants n, g, g_1 in state space that are perspectively equivalent to a given one. This problem has been discussed in Section IV.

In Section V we introduce an algorithm to identify parameters of a perspective system up to orbits of the perspective group. This newly introduced algorithm is called "the rescaling algorithm" and signifies the fact that an observed sequence of nonzero vectors have to be rescaled by an unknown sequence of nonzero scalars before it can be realized as an output of a dynamical system of degree n. The rescaling algorithm computes the unknown scalar sequence.

In Sections VI–VIII, as an application of the proposed realization theory described in Sections II–V to problems in machine vision, we consider a planar surface undergoing an affine motion and assume that the surface is observed by a CCD camera. The problem of motion and shape identification in this context is shown equivalent to parameter identification of a perspective system. A complete set of identifiable motion and shape parameters has been characterized when the camera-observation function is assumed to be "perspective" and "orthographic" projections. We also analyze the special case when the planar surface undergoes a rigid motion and recovers many known results in the literature [1], [2].

In summary, we introduce a new theory of perspective dynamical system and apply the theory to motion and shape estimation problems in machine vision.

II. The Perspective Observability Problem

In this section we propose to consider Problem 1.3 but introduce and analyze the problem in slightly more generality. We shall let $\mathbb{K}$ denote either the field of real ($\mathbb{K} = \mathbb{R}$) or the field of complex ($\mathbb{K} = \mathbb{C}$) numbers. Let A be an $n \times n$ matrix and C be a $p \times n$ matrix defined over $\mathbb{K}$. We consider the linear time invariant system (10), where $x_k \in \mathbb{K}^n, z_k \in \mathbb{K}^p$. Recall that the well-known Hautus's test [10] gives a necessary and sufficient condition when the state vector x_k can be observed from the output measurement z_k. To be precise, one has the following.

Theorem 2.1 (Hautus [10]): System (10) is observable over either $\mathbb{R}$ or $\mathbb{C}$ if and only if

$$\operatorname{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{15}$$

To introduce the main result considered in this section, let $P_0 \subset \mathbb{K}^n$ be a d-dimensional plane not necessarily passing through the origin. We say that dynamical system (10) is perspectively observable with respect to d-dimensional planes if for every pair of d-planes P_0, Q_0 such that $P_0 \neq Q_0$ there exist a $k \geq 0$ such that CA^kP_0 and CA^kQ_0 are both d-planes and $CA^kP_0 \neq CA^kQ_0$. Our main result is described as follows.

Theorem 2.2: System (10) is observable with respect to d-dimensional planes in $\mathbb{K}^n$ if for any set of eigenvalues $\lambda_0, \cdots, \lambda_d$ of A one has

$$\operatorname{rank}\begin{pmatrix} (A - \lambda_0 I)(A - \lambda_1 I) \cdots (A - \lambda_d I) \\ C \end{pmatrix} = n. \tag{16}$$

Moreover, this condition is also necessary if $d = 0$ or if the eigenvalues of the matrix A are in $\mathbb{K}$.

Note in particular that Theorem 1.4 is a special case of Theorem 2.2 when $d = 1$. In fact, Theorem 2.2 follows from an immediate adaptation of a similar result reported in [8] for a continuous-time system. Only an essence of the proof is therefore sketched below. The proof of Theorem 2.2 is based on a careful study of a dynamical system defined on the vector space $\wedge^{d+1}\mathbb{K}^n$, the $(d+1)$-fold wedge product of $\mathbb{K}^n$ [11]. Define linear maps as follows:

$$\begin{aligned}\hat{A} : \wedge^{d+1}\mathbb{K}^n &\rightarrow \wedge^{d+1}\mathbb{K}^n\\ x_0\wedge\cdots\wedge x_d &\mapsto Ax_0\wedge\cdots\wedge Ax_d\end{aligned} \tag{17}$$

and

$$\begin{aligned}\hat{C} : \wedge^{d+1}\mathbb{K}^n &\rightarrow \wedge^{d+1}\mathbb{K}^p\\ x_0\wedge\cdots\wedge x_d &\mapsto Cx_0\wedge\cdots\wedge Cx_d.\end{aligned} \tag{18}$$

Using the above two maps, we consider the dynamical system

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k \quad \hat{y}_k = \hat{C}\hat{x}_k. \tag{19}$$

It can be shown that provided the eigenvalues of A are in $\mathbb{K}$, (16) is equivalent to a particular notion of observability of system (19), and this condition is also necessary and sufficient for the observability of P_0 under the output function CA^kP_0, i.e., perspective observability of (10) with respect to d-dimensional planes.

The statement of the main result which is an easy adaptation of the result described in [7] and [8] is now stated.

Lemma 2.3 (Main Lemma): Assume that the eigenvalues of the matrix A are in $\mathbb{K}$. Then the following conditions are equivalent.

1) There are eigenvalues $\lambda_{i_0},\cdots,\lambda_{i_d}$ of A and a nonzero vector $v\in\mathbb{K}^n$

$$\begin{bmatrix}(A-\lambda_{i_0}I)\cdots(A-\lambda_{i_d}I)\\ C\end{bmatrix}v=0.$$

2) There is a $\lambda\in\mathbb{K}$ and a decomposable vector $\beta_0\wedge\cdots\wedge\beta_d\in\wedge^{d+1}\mathbb{K}^n$

$$\begin{pmatrix}\hat{A}-\lambda I\\ \hat{C}\end{pmatrix}\beta_0\wedge\cdots\wedge\beta_d=0.$$

3) Dynamical system (19) has a decomposable vector $\alpha_0\wedge\cdots\wedge\alpha_d\in\wedge^{d+1}\mathbb{K}^n$ in its unobservable subspace.

Proof 1) $\mapsto$ 2): Let v be a vector in the kernel of the matrix

$$\begin{bmatrix}(A-\lambda_{i_0}I)\cdots(A-\lambda_{i_d}I)\\ C\end{bmatrix}.$$

It follows that v can be decomposed as

$$v=v_0+\cdots+v_d$$

where v_s is in the eigenspace of λ_{i_s}. Furthermore, it can be deduced from [8] that

$$v_0\wedge\cdots\wedge v_d$$

is an eigenvector of $\hat{A}$ corresponding to the eigenvalue $\lambda=\lambda_{i_0}\lambda_{i_1}\cdots\lambda_{i_d}$. Furthermore, since $Cv=0$ it follows that $\hat{C}(v_0\wedge\cdots\wedge v_d)=0$.

2) $\mapsto$ 3): The vector $\beta_0\wedge\cdots\wedge\beta_d$ is necessarily an eigenvector of $\hat{A}$ and is therefore in the unobservable subspace U of (19).

3) $\mapsto$ 1): This case is nontrivial but follows from an application of [7] or [8]. The details are omitted.

Lemma 2.3 is similar but not exactly what has been proved in [7] and [8]. The difference lies in the fact that the linear map $\hat{A}$ in (17), is different from that defined in [7] and [8].

We now proceed to prove Theorem 2.2 as follows.

Proof of Theorem 2.2 (Sufficiency): Let $P_0,Q_0\subset\mathbb{K}^n$ be two d-dimensional planes with $P_0\neq Q_0$. Let $q_0\in Q_0$ be a point such that $q_0\notin P_0$. Let $\{x_0,\cdots,x_d\}\subset P_0$ be a set of points in general position, and consider the nonzero decomposable vector

$$w\triangleq(q_0-x_0)\wedge(x_1-x_0)\wedge\cdots\wedge(x_d-x_0).$$

If (16) holds, it follows from Lemma 2.3 that there is no decomposable vector in the unobservable subspace U of (19). Thus $w\notin U$. Hence $\hat{C}\hat{A}^kw\neq0$ for some $k\geq0$. It follows that

$$\begin{aligned}(CA^kq_0-CA^kx_0)\wedge(CA^kx_1-CA^kx_0)\wedge\cdots\cdots\\ \wedge(CA^kx_d-CA^kx_0)\end{aligned}$$

is nonzero for some $k\geq0$. But then we have that $CA^kq_0\notin CA^kP_0$ for some $k\geq0$, i.e., $CA^kQ_0\neq CA^kP_0$ for some $k\geq0$.

(Necessity): Assume that there is a set of eigenvalues $\lambda_0,\cdots,\lambda_d$ of A such that the rank condition (16) is not satisfied. By Lemma 2.3 it follows that there exists a nonzero decomposable vector $x_0\wedge\cdots\wedge x_d$ in the unobservable subspace U of (19).

Define $V=\operatorname{span}\{x_0,\cdots,x_d\}$. Clearly V is a $d+1$ dimensional subspace. We now define a sequence of subspaces

$$U_k\triangleq\operatorname{span}\{CA^kx_0,\cdots,CA^kx_d\}$$

$k=0,1,2,\cdots$. Since $CA^kx_0\wedge\cdots\wedge CA^kx_d$ is a zero vector for all $k\geq0$, it follows that dim $U_k\leq d$ for all $k\geq0$. We now define a map

$$\begin{aligned}\Psi_k: V&\rightarrow U_k\\ x&\mapsto CA^kx.\end{aligned}$$

Let us now consider the set of all d-dimensional subspaces of V (call it V_d) such that Ψ_k restricted to V_d is surjective. It follows that in Grass $(d,d+1)$ the set of all such subspaces form an open and dense set $\mathcal{K}_k$ for each k. Define $\mathcal{K}=\cap_k\mathcal{K}_k$, where $\mathcal{K}$ is clearly dense in Grass $(d,d+1)$. Let P_0 and Q_0 be two distinct points in $\mathcal{K}$. It follows that

$$CA^kP_0=CA^kQ_0=U_k$$

for $k\geq0$. Thus P_0 and Q_0 cannot be observed. ∎

To summarize, in this section we have considered the discrete-time system (10) under the hypothesis that z_k is observed up to a d-dimensional plane. The problem that we have considered is to observe the initial condition x_0 up to a d-dimensional plane. We have obtained a generalization (16) of the Hautus's observability criterion (15). Of course the observability problem presupposes that the parameters A and C are known. If this is not the case (as would typically be in various problems of machine vision), then one needs to consider the perspective identification Problem 1.6, described in Sections III–V.

III. The Perspective Identifiability Problem

The purpose of this section is to analyze Problem 1.6, the perspective identifiability problem, and in so doing prove Theorem 1.7. We start off by considering the linear dynamical system (10) together with the observation function (11) and show that if the parameters $A, C, [x_0]$ are unknown, it is possible to recover the parameters up to an orbit of a perspective group.

Assume that (10) is minimal as a linear dynamical system with state dimension n. Let us consider the output sequence

$$\{z_0, z_1, z_2, \cdots\} \tag{20}$$

of p-vectors and define

$$H(z_j) = \begin{pmatrix} z_0 & z_1 & z_2 & \cdot & \cdot & \cdot \\ z_1 & z_2 & z_3 & \cdot & \cdot & \cdot \\ z_2 & z_3 & z_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{21}$$

to be the Hankel matrix corresponding to (20). Clearly rank $H(z_j) = n$. Thus there exists $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ such that

$$z_{j+n} = \alpha_0 z_j + \alpha_1 z_{j+1} + \cdots + \alpha_{n-1} z_{j+n-1} \tag{22}$$

for $j = 0, 1, \cdots$. We now assume that (20) is such that at least one of the real numbers $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ is nonzero. The alternative case is trivial and will be dealt with later on.

We now define two additional integer invariants g, g_1 that correspond to (20). Let g_1 be the smallest integer such that $\alpha_{g_1} \neq 0$. We now define a set S of integers as follows:

$$S = \{k - g_1 : \alpha_k \neq 0, g_1 < k \leq n-1\} \cup \{n - g_1\}. \tag{23}$$

The integer g is defined to be the greatest common factor (g.c.f.) of all elements in S. We have thus associated three unique integers n, g_1, g to (20). Let us now make the following important assumption about sequence (20).

Assumption 3.1: The vectors

$$\{z_{g_1+j}, z_{g_1+g+j}, \cdots, z_{g_1+(\theta-1)g+j}\} \tag{24}$$

are linearly independent for all $j = 0, 1, \cdots, g-1$, where $\theta = (n - g_1)/g$.

Note that (22) can be written equivalently as

$$\begin{aligned} z_{j+n} =& \alpha_{g_1} z_{j+g_1} + \alpha_{g_1+g} z_{j+g_1+g} + \alpha_{g_1} \\ &+ (\theta - 1)g\, z_{j+g_1+(\theta-1)g}. \end{aligned} \tag{25}$$

An important consequence of Assumption 3.1 is described as follows. Let us consider a sequence

$$\{y_0, y_1, y_2, \cdots\} \tag{26}$$

of p-dimensional vectors and assume that for some nonzero sequence

$$\{\delta_0, \delta_1, \delta_2, \cdots\} \tag{27}$$

of scalars, we have

$$y_j = \delta_j z_j, \quad j = 0, 1, 2, \cdots \tag{28}$$

where z_j is the sequence (20) that satisfies Assumption 3.1.

Definition 3.2: We shall say that sequence (26) preserves the integers n, g, g_1 (where $g_1 < n$ and g divides $n - g_1$) if there exists $\beta_{g_1}, \beta_{g_1+g}, \beta_{g_1+2g}, \cdots, \beta_{g_1+(\theta-1)g}$ such that for $j = 0, 1, \cdots$ we have

$$\begin{aligned} y_{j+n} =& \beta_{g_1} y_{j+g_1} + \beta_{g_1+g} y_{j+g_1+g} \\ &+ \cdots + \beta_{g_1+(\theta-1)g} y_{j+g_1+(\theta-1)g} \end{aligned} \tag{29}$$

where $\theta = (n - g_1)/g$.

We now state the main result of this paper.

Theorem 3.3 (Main Theorem): Let us assume that we are given sequence (20) of p-dimensional vectors satisfying Assumption 3.1. Additionally we assume that (20) is such that at least one of the real numbers $\alpha_0, \cdots, \alpha_{n-1}$ is nonzero so that one can uniquely define integers n, g, and g_1. Let (26) be any other sequence satisfying (28) for some nonzero sequence (27), then the following two conditions are equivalent.

1) Sequence (26) preserves the integers n, g, g_1 corresponding to sequence (20).
2) Sequence (27) satisfies the condition

$$\delta_{g_1} \delta_{j+g} = \delta_j \delta_{g+g_1} \tag{30}$$

for all $j = g_1 + 1, g_1 + 2, \cdots$.

Furthermore, if (29) is satisfied, then

$$n \geq \operatorname{rank} H(y_j) \geq \frac{n - g_1}{g}. \tag{31}$$

We now make the following two remarks.

Remark 3.4: Note that the sequence (20) is of Hankel rank n, and in fact the integers n, g, g_1 are uniquely defined for sequence (20). Sequence (26) derived from (20) is not necessarily of Hankel rank n. Thus to say that (26) preserves the integers n, g, and g_1 does not mean that the choice of these integers is unique.

Remark 3.5: In general, for an arbitrary scale factor sequence (27), the Hankel rank of (26) could be arbitrary large. The point of the Main Theorem 3.3 is that under a suitable recursion (20), sequence (26) preserves the integers n, g, g_1, i.e., it satisfies recursion (29). More importantly, the Hankel rank of (26) is upper bounded by n and lower bounded by $(n - g_1)/g$.

In view of (29), note that there are exactly $g + g_1 + 1$ arbitrary elements in the scale factor sequence (27). Thus we have a group Gp defined as follows:

$$G_p \triangleq \underbrace{\mathbb{R}^* \times \mathbb{R}^* \times \cdots \times \mathbb{R}^*}_{\rightarrow g+g_1+1 \text{ fold} \leftarrow} \tag{32}$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$. Note that $\mathbb{R}^*$ is a group under multiplication and G_p is a group under component wise multiplication. Let S be the space of sequence of p-vectors. We now define the action of G_p on S as follows:

$$\begin{aligned} \pi \colon\ & G_p \times S \to S \\ & ((\delta_0, \cdots, \delta_{g+g_1}), (z_0, z_1, \cdots, z_{g+g_1}, \cdots)) \\ & \mapsto (\delta_0 z_0, \delta_1 z_1, \cdots, \delta_{g+g_1} z_{g+g_1}, \cdots) \end{aligned} \tag{33}$$

where δ_j's satisfy recursion (30). The following theorem describes the main identifiability result.

Theorem 3.6: Let $\{\xi_j\}$ and $\{\rho_j\}$ be two sequences of p-dimensional vectors that preserve integers n, g, g_1 and satisfy Assumption 3.1. The following two conditions are equivalent.

1) $\{\xi_j\}$ and $\{\rho_j\}$ are in the same orbit of the G_p action described in (33).
2) There exists a sequence $\{\delta_0, \delta_1, \cdots\}$ of nonzero scalars such that $\xi_j = \delta_j \rho_j, j = 0, 1, \cdots$.

Remark 3.7: It may be remarked that the sequences $\{\xi_j\}$ and $\{\rho_j\}$ are perspectively indistinguishable whenever $G_j = \delta_j \rho_j, j = 0, 1, \cdots$. This is because, up to a scale factor, ξ_j and ρ_j are the same vectors for every $j \geq 0$. Theorem 3.6 provides a characterization of perspectively indistinguishable sequences as an orbit of a group Gp action.

We shall now present a proof of the Main Theorem 3.3, first of all by considering the special case when $g_1 = 0$. We also assume $\delta_0 = 1$ without any loss of generality. We have the following lemma.

Lemma 3.8: Let us consider a sequence (30) of Hankel rank "n" with associated integers n, g_1, g. Assume $g_1 = 0$ and let (30) satisfy Assumption 3.1. Let (26) be any other sequence which satisfies (28) for some nonzero sequence (27), where we assume $\delta_0 = 1$. It follows that (26) preserves the integers n, g, g_1 corresponding to (30) if and only if for all $j = 0, 1, 2, \cdots$

$$\delta_{j+g} = \delta_j \delta_g. \tag{34}$$

To prove Lemma 3.8, we begin with a definition and some simple propositions.

Definition 3.9: Let $\Delta = \{\delta_j\}_{j=0}^{\infty}$ be a sequence of nonzero real numbers with $\delta_0 = 1$. We say that the integer $k > 0$ is a generator for Δ if for all $j = 0, 1, 2, \cdots$ we have

$$\delta_{j+k} = \delta_j \delta_k. \tag{35}$$

Proposition 3.10: If k is a generator for Δ, then for all $j = 0, 1, 2, \cdots$

$$\delta_{jk} = \delta_k^j. \tag{36}$$

The proof is clear.

Proposition 3.11: If k is a generator for Δ, then for any $m > 0, \delta_m$ may be written as a product of $\delta_1, \delta_2, \cdots, \delta_k$.

Proof: Using (35) and (36), we may write

$$\delta_m = \delta_k^j \delta_s \tag{37}$$

where $m = jk + s, j \geq 0$, and $s < k$. ■

Proposition 3.12: The integers k_1 and k_2 are both generators for Δ if and only if their g.c.f. is a generator for Δ.

Proof: Sufficiency is clear since k_1 and k_2 are both multiples of their g.c.f. g, and thus for all $j = 0, 1, 2, \cdots$ we have

$$\delta_{j+k_1} = \delta_{j+ng} = \delta_j \delta_g^n = \delta_j \delta_{ng} = \delta_j \delta_{k_1}. \tag{38}$$

Likewise one can write a similar statement for k_2. To show necessity, we suppose without any loss of generality that $k_1 > k_2$ and write

$$k_1 = nk_2 + s \tag{39}$$

where $s < k_2$ (note that if $k_1 = k_2$, the proof is trivial). We claim that s is also a generator for Δ. Now since k_2 is a generator for Δ, by Proposition 3.11 and from (39) we may write

$$\delta_{k_1} = \delta_{k_2}^n \delta_s. \tag{40}$$

Since k_1 is a generator for Δ, from (40) we have for all $j = 0, 1, 2, \cdots$

$$\delta_{j+k_1} = \delta_j \delta_{k_1} = \delta_j \delta_{k_2}^n \delta_s. \tag{41}$$

On the other hand, since k_2 is a generator for Δ, from (39) we have

$$\delta_{j+k_1} = \delta_{j+nk_2+s} = \delta_{j+s} \delta_{k_2}^n. \tag{42}$$

Comparing (41) and (42) we have

$$\delta_{j+s} = \delta_j \delta_s \tag{43}$$

and thus s is a generator for Δ. If $s = 0$, then k_2 is the g.c.f. If $s \neq 0$, we may repeat the above argument replacing k_1 and k_2 with k_2 and s, respectively. Such recursion will conclude with $s = 0$, in which case k_2 is the desired g.c.f. ■

We now prove Lemma 3.8.

Proof (Sufficiency): To show that (26) preserves the integers n, g, and g_1, we must find $\beta_0, \beta_g, \beta_{2g}, \cdots, \beta_{n-g}$ such that for all $j = 0, 1, 2, \cdots$

$$\begin{aligned} \delta_{j+n} z_{j+n} = &\beta_0 \delta_j z_j + \beta_g \delta_{j+g} z_{j+g} \\ &+ \cdots + \beta_{n-g} \delta_{j+n-g} z_{j+n-g}. \end{aligned} \tag{44}$$

We claim that such a choice of β_k is given by

$$\beta_k = \alpha_k \frac{\delta_{j+n}}{\delta_{j+k}} \tag{45}$$

for $k = 0, g, \cdots, n - g$. Notice first of all that from (25) and (45), (44) follows. What remains to be shown is that β_k as defined via (45) is independent of j. ■

Note that every k in the set $\{g, \cdots, n-g, n\}$ is a multiple of g. Hence they are all generators of Δ. Thus we conclude that

$$\delta_{j+n} = \delta_j \delta_n \quad \text{and} \quad \delta_{j+k} = \delta_j \delta_k.$$

It follows that (45) can actually be written as

$$\beta_k = \alpha_k \frac{\delta_n}{\delta_k} \tag{46}$$

which is clearly independent of j.

(Necessity): We assume that (26) preserves the integers n, g, and g_1 associated with the sequence $\{z_j\}_{j=0}^{\infty}$, i.e., that there exist $\beta_0, \beta_g, \beta_{2g}, \cdots, \beta_{n-g}$ satisfying (44). Let us rewrite (44) as

$$\begin{aligned} z_{j+n} = &\beta_0 \frac{\delta_j}{\delta_{j+n}} z_j + \beta_g \frac{\delta_{j+g}}{\delta_{j+n}} z_{j+g} + \cdots + \beta_{n-g} \frac{\delta_{j+n-g}}{\delta_{j+n}} \\ &\cdot z_{j+n-g}. \end{aligned} \tag{47}$$

Since (30) satisfies Assumption 3.1, it also follows that the coefficients $\alpha_0, \alpha_g, \alpha_{2g}, \cdots, \alpha_{n-g}$ in (25) are unique. Comparing (47) and (25), it follows that

$$\beta_0 \frac{\delta_j}{\delta_{j+n}}, \quad \beta_g \frac{\delta_{j+g}}{\delta_{j+n}}, \quad \beta_{n-g} \frac{\delta_{j+n-g}}{\delta_{j+n}} \tag{48}$$

are all independent of j. Note that by the definition of g_1, β_0 is nonzero since $g_1 = 0$. However, in general β_{kg} for certain $k = 1, 2, \cdots, \theta - 1$ can be 0. If we assume β_g is nonzero, we may proceed as follows. Since δ_j/δ_{j+n} is independent of j, it follows that

$$\delta_{n+j} = \delta_n \delta_j \tag{49}$$

assuming $\delta_0 = 1$. On the other hand, since $\delta_{j+g}/\delta_{j+n}$ is independent of j, it follows that

$$\delta_{n+j}\delta_g = \delta_n \delta_{g+j}. \tag{50}$$

Combining (49) and (50) we have

$$\delta_{g+j} = \delta_g \delta_j$$

for $j = 0, 1, 2, \cdots$. Hence g is a generator of Δ. (Q.E.D.)

Analogously one can show that for all k such that $\beta_{kg} \neq 0, kg$ is a generator of Δ. Since by definition g is the g.c.f. of all kg such that $\beta_{kg} \neq 0, k = 1, 2, 3, \cdots$, it would follow that in general g is a generator of Δ.

Remark 3.13: If in fact we assume that $\delta_0 \neq 1$, then Lemma 3.8 can be restated with (34) replaced by

$$\delta_0 \delta_{j+g} = \delta_j \delta_g \tag{51}$$

for $j = 0, 1, 2, \cdots$.

Proof of the Main Theorem 3.3: Let us first assume that $g_1 = 0$. In this case, (30) reduces to (51) which is equivalent to (34) whenever $\delta_0 = 1$. Thus from Lemma 3.8, the equivalence of Conditions 1) and 2) of Theorem 3.3 is verified.

Note that the upper bound on the rank of $\mathcal{H}(y_j)$ is satisfied because the integer n is preserved. On the other hand by Assumption 3.1, $\mathcal{H}(z_j)$ has at least n/g linearly independent columns. This is because $z_0, z_g, \cdots, z_{n-g}$ are linearly independent vectors. It follows that $y_0, y_g, \cdots, y_{n-g}$ also remain linearly independent and hence $\mathcal{H}(y_j)$ has at least n/g linearly independent columns.

When $g_1 \neq 0$, one can reparameterize all sequences to start from $j = g_1$ instead of $j = 0$. Thus we consider a new sequence $\{z_j\}, j = g_1, g_1 + 1, \cdots$ and $\{y_j\}, j = g_1, g_1 + 1, \cdots$ and prove a lemma analogous to Lemma 3.8. Condition (51) would be modified as (30) verifying the equivalence of 1) and 2).

Once again, the upper bound on the rank of $H(y_j)$ is satisfied because the integer n is preserved. On the other hand, by Assumption 3.1, $H(y_j)$ has at least θ independent columns. This is because the vectors $z_{g_1}, z_{g_1+g}, \cdots, z_{g_1+(\theta-1)g}$ are linearly independent. Hence the vectors $y_{g_1}, y_{g_1+g}, \cdots, y_{g_1+(\theta-1)g}$ are also linearly independent. Hence

$$\operatorname{rank} \mathcal{H}(y_j) \geq \theta = (n - g_1)/g.$$

(Q.E.D.)

We now proceed to prove Theorem 3.6.

Proof of Theorem 3.6: Let $\{\xi_j\}$ and $\{\rho_j\}$ be two sequences of p-dimensional vectors that preserve n, g, g_1 and satisfy Assumption 3.1. If 1) is satisfied, clearly

$$\rho_j = \delta_j \xi_j$$

for some nonzero sequence $\{\delta_0, \delta_1, \cdots\}$. Hence 2) is satisfied. Conversely if 2) is satisfied, we have two sequences $\{\xi_j\}$ and $\{\rho_j\}$ that preserve integers n, g, g_1 and satisfy the assumptions of the Main Theorem 3.3. Hence it follows that the sequence $\{\delta_0, \delta_1, \cdots\}$ satisfies (30), i.e., $\{\xi_j\}$ and $\{\rho_j\}$ are in the same G_p orbit. (Q.E.D.)

As a final result of this section, we prove Theorem 1.7 stated in the introduction.

Proof of Theorem 1.7: This theorem is a special case of the Main Theorem 3.3 when $g_1 = 0, g = 1$. In this case (30) reduces to

$$\delta_0 \delta_{j+1} = \delta_j \delta_1 \tag{52}$$

for $j = 1, 2, \cdots$. Sequence (27) reduces to

$$\{\delta_0, \delta_1, \delta_1^2/\delta_0, \delta_1^3/\delta_0^2, \cdots\}.$$

Thus every sequence which preserves $n, g = 1, g_1 = 0$ and is perspectively indistinguishable from the sequence

$$\{Cx_0, CAx_0, \cdots\} \tag{53}$$

must be of the form

$$\left\{\delta_0 Cx_0, \delta_0 C\left(\frac{\delta_1}{\delta_0}A\right)x_0, \delta_0 C\left(\frac{\delta_1}{\delta_0}A\right)^2 x_0, \cdots\right\}. \tag{54}$$

Thus if the triplet (C, A, x_0) generates the sequence (53), one obtains that the triplet $(\lambda_1 C, \lambda_2 A, \lambda_3 x_0)$ would generate (54) provided $\lambda_1\lambda_3 = \delta_0, \lambda_2 = \delta_1/\delta_0$. Thus we have the group action described by (14). Finally, since the triplet (C, A, x_0) is minimal as a linear dynamical system, we have the action described by (13). (Q.E.D.)

Remark 3.14: As a final remark of this section, we comment on the case when $\alpha_0, \cdots, \alpha_{n-1}$ are all 0 in (22). In this case, (20) reduces to

$$\{z_0, z_1, \cdots, z_{n-1}, 0, \cdots, 0\}. \tag{55}$$

It is trivial to verify that every other sequence which is perspectively indistinguishable from (20) must be of the form

$$\{\delta_0 z_0, \delta_1 z_1, \cdots, \delta_{n-1} z_{n-1}, 0, \cdots, 0\} \tag{56}$$

for arbitrary nonzero scalars $\delta_0, \cdots, \delta_{n-1}$.

IV. State-Space Realization of Perspective Systems

In this section, we shall study the following problem.

Problem 4.1 (Perspective Realization Problem): Consider the perspective dynamical system (10), (11) and assume that we observe $[z_k]$, for $k = 0, 1, 2, \cdots$, except perhaps when $z_k = 0$. Assume furthermore that (10) is of minimal state dimension n as a linear dynamical system and that the sequence $z_j, j = 0, 1, 2, \cdots$ satisfies Assumption 3.1. The problem is to characterize the set of all realizations (A, C, x_0) that produce the same perspective output $[z_k]$ with the state dimension upper bounded by n. Furthermore, we require that if the sequence $\{z_k\}_{k=0}^{\infty}$ corresponds with the integer g, g_1, then for every realization (A, C, x_0) the sequence $Cx_0, CAx_0, \cdots$ also preserves the integers g, g_1.

Remark 4.2: Recall that if instead of $[z_k]$ one observes z_k for $k = 0, 1, 2, \cdots$, then the set of realizations (A, C, x_0) of minimal state dimension n that would produce the output z_k can be characterized as follows. If (A, C, x_0) is one realization of minimal state dimension n, then the set of all other realizations of state dimension n is given by $(P^{-1}AP, CP, P^{-1}x_0)$, where P is any $n \times n$ nonsingular matrix. A similar characterization for perspective systems is detailed in this paper.

Recall that if (20) is a sequence of rank n satisfying the recursive relation (22), one can construct a state-space system of the form (10) with impulse response given exactly as (20). In particular, one can obtain

$$x_0 = (0, 0, \cdots, 0, 1)^T \tag{57}$$

$$C = (c_1, c_2, \cdots, c_n) \tag{58}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{pmatrix}. \tag{59}$$

Moreover, every other realization which realizes (20) is given by $(P^{-1}AP, CP, P^{-1}x_0)$, where P is any $n \times n$ nonsingular matrix. The matrix C in (58) is computed as

$$C = (z_0, z_1, \cdots, z_{n-1})(x_0, Ax_0, \cdots, A^{n-1}x_0)^{-1}. \tag{60}$$

If (20) is scaled by the nonzero sequence (27), provided that (30) is satisfied, the new sequence (26) can be realized as

$$x_0' = (0, 0, \cdots, 0, 1)^T \tag{61}$$

$$C' = (c_1', c_2', \cdots, c_n') \tag{62}$$

and

$$A' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_0 \dfrac{\delta_n}{\delta_0} & \alpha_1 \dfrac{\delta_n}{\delta_1} & \alpha_2 \dfrac{\delta_n}{\delta_2} & \cdots & \alpha_{n-1} \dfrac{\delta_n}{\delta_{n-1}} \end{pmatrix}. \tag{63}$$

The matrix C' in (62) can be computed as

$$C' = (\delta_0 z_0, \delta_1 z_1, \cdots, \delta_{n-1} z_{n-1}) \cdot (x_0', A' x_0', \cdots, A'^{n-1} x_0')^{-1}. \tag{64}$$

The triplet (61)–(63) characterizes the set of all controllable realizations (of state dimension n) of sequences that are perspectively indistinguishable from (20) and preserves the integers g, g_1. As has been remarked before, the triplet is not necessarily minimal.

In particular if $g_1 = 0$ and $g = 1$, the matrix A' is as described in (63) with the last row given by

$$\left(\alpha_0 \left(\frac{\delta_1}{\delta_0} \right)^n, \alpha_1 \left(\frac{\delta_1}{\delta_0} \right)^{n-1}, \cdots, \alpha_{n-1} \left(\frac{\delta_1}{\delta_0} \right) \right). \tag{65}$$

Matrix C in (62) is given by

$$C' = \left(\delta_0 z_0, \delta_1 z_1, \frac{\delta_1^2}{\delta_0} z_2, \cdots, \frac{\delta_1^{n-1}}{\delta_0^{n-2}} z_{n-1} \right) \cdot (x_0', A' x_0', \cdots, A'^{n-1} x_0')^{-1}. \tag{66}$$

The triplet (61), (63), (66), with the last row of "A'" replaced by (65), is the set of all controllable realizations (of state dimension n) of sequences that are perspectively indistinguishable from (20) while preserving parameters $g = 1$ and $g_1 = 0$. Note that the controllable realizations are parameterized by exactly two parameters δ_0 and δ_1.

In general, for arbitrary g_1 and g, the last row of the matrix "A'" can be written as follows. Assume that $n = g_1 + \theta g$, then the last row is given by

$$\Bigg(\underbrace{0, 0, \cdots, 0,}_{g_1} \underbrace{\left(\frac{\delta_{g_1+g}}{\delta_{g_1}} \right)^{\theta} \alpha_{g_1}, 0, \cdots, 0,}_{g} \left(\frac{\delta_{g_1+g}}{\delta_{g_1}} \right)^{\theta-1} \alpha_{g_1+g}, 0, \cdots, 0, \; \cdots\cdots\cdots \; \cdots\cdots\cdots \left(\frac{\delta_{g_1+g}}{\delta_{g_1}} \right) \alpha_{g_1+(\theta-1)g}, 0, \cdots, 0 \Bigg). \tag{67}$$

The matrices x_0' and C' are given, respectively, by (61) and (64), where the vector $(\delta_0, \cdots, \delta_{n-1})$ depends on $g + g_1 + 1$ free parameters $(\delta_0, \cdots, \delta_{g+g_1})$, as is clear from (30).

Thus to sum up, if a sequence (20) is of Hankel rank n and corresponds to integers g and g_1, and if the sequence satisfies Assumption 3.1, then the set of all controllable realizations of sequences that are perspectively indistinguishable from (20) is characterized by $g + g_1 + 1$ free parameters.

V. A Rescaling Algorithm

In this section, we consider a sequence of p-dimensional vectors $z_j, j = 0, 1, \cdots$. We assume furthermore that there exist three integers n, g, g_1 where $g_1 < n$, and g divides $n - g_1$ with respect to which the sequence $\{z_j\}$ satisfies Assumption 3.1. We assume furthermore that the sequence $\{z_j\}$ is not observed. Rather, the observed sequence of vectors is given by $\{q_j\}$ where

$$q_j = \delta_j z_j \tag{68}$$

where the δ_j's are a sequence of arbitrary nonzero scalars. We once again assume that (C, A, x_0) is a minimal triplet which generates the sequence $\{z_j\}$ and thus $\operatorname{rank}[\mathcal{H}(z_j)] = n$. To rescale the observed output sequence $\{q_j\}_{j=0}^{\infty}$, we wish to find a sequence of nonzero scalars $\lambda_0, \lambda_1, \lambda_2, \cdots$ such that for the sequence $\{\lambda_j q_j\}_{j=0}^{\infty}$, the integers n, g_1, and g are preserved. It follows that there exist scalars $\beta_{g_1}, \beta_{g_1+g}, \cdots, \beta_{g_1+(\theta-1)g}$ such that

$$\begin{aligned}\lambda_{j+n} q_{j+n} = \beta_{g_1} q_{g_1+j} \lambda_{g_1+j} + \beta_{g_1+g} q_{g_1+g+j} \lambda_{g_1+g+j} \\ + \beta_{g_1+2g} q_{g_1+2g+j} \lambda_{g_1+2g+j} \cdots \\ + \beta_{n-g} q_{n-g+j} \lambda_{n-g+j}\end{aligned} \tag{69}$$

for $j = 0, 1, 2, \cdots$ and where $g_1 + \theta g = n$. By Assumption 3.1, we have that

$$q_{g_1+j}, q_{g_1+j+g}, \cdots, q_{g_1+j+(\theta-1)g} = q_{n+j-g}$$

are independent vectors. Therefore, one can compute the coefficients

$$\beta_{g_1} \frac{\lambda_{g_1+j}}{\lambda_{n+j}}, \beta_{g_1+g} \frac{\lambda_{g_1+j+g}}{\lambda_{n+j}}, \cdots, \beta_{n-g} \frac{\lambda_{n+j-g}}{\lambda_{n+j}} \tag{70}$$

uniquely from (69). Let us denote the vector in (70) as

$$\alpha_{g_1,j}, \alpha_{g_1+g,j}, \alpha_{g_1+2g,j}, \cdots, \alpha_{g_1+(\theta-1)g,j}. \tag{71}$$

From (70) and (71) it follows that

$$\beta_{g_1} = \frac{\lambda_n}{\lambda_{g_1}} \alpha_{g_1,0} = \frac{\lambda_{n+j}}{\lambda_{g_1+j}} \alpha_{g_1,j} \tag{72}$$

$$\beta_{g_1+g} = \frac{\lambda_n}{\lambda_{g_1+g}} \alpha_{g_1+g,0} = \frac{\lambda_{n+j}}{\lambda_{g_1+g+j}} \alpha_{g_1+g,j} \tag{73}$$

for $j = 0, 1, 2, \cdots$. Finally by comparing (72) and (73) we have

$$\frac{\lambda_n}{\lambda_{n+j}} = \frac{\lambda_{g_1}}{\lambda_{g_1+j}} \frac{\alpha_{g_1,j}}{\alpha_{g_1,0}} = \frac{\lambda_{g_1+g}}{\lambda_{g_1+g+j}} \frac{\alpha_{g_1+g,j}}{\alpha_{g_1+g_1,0}}. \tag{74}$$

It follows that

$$\lambda_{g_1+g+j} = \frac{\lambda_{g_1+g} \lambda_{g_1+j}}{\lambda_{g_1}} \frac{\alpha_{g_1,0}}{\alpha_{g_1,j}} \frac{\alpha_{g_1+g,j}}{\alpha_{g_1+g,0}}. \tag{75}$$

The formula (75) describes the key formula for the rescaling algorithm. Note that from the definition of g_1, it follows that $\alpha_{g_1,j} \neq 0$ for all j. However, $\alpha_{g_1+g,0}$ can be zero and (75) is applicable when $\alpha_{g_1+g,0} \neq 0$.

In the case when $\alpha_{g_1+g,0}$ is zero, we encounter a singular case, and (75) is no longer applicable. For notational simplicity, we shall describe the new recursion assuming $g_1 = 0$. Rewriting (70), it follows that one can compute the coefficients

$$\beta_0 \frac{\lambda_j}{\lambda_{n+j}}, \beta_g \frac{\lambda_{j+g}}{\lambda_{j+n}}, \beta_{2g} \frac{\lambda_{j+2g}}{\lambda_{j+n}}, \cdots, \beta_{n-g} \frac{\lambda_{j+n-g}}{\lambda_{j+n}} \tag{76}$$

uniquely from (69). Note that the singular case corresponds precisely to the case when $\beta_g = 0$. We shall now assume that $\beta_{k_1} \neq 0$ and $\beta_{k_2} \neq 0$, where k_1 and k_2 are some multiples of g.

If $\beta_{k_1} \neq 0$, it follows that

$$\frac{\lambda_n}{\lambda_{n+j}} = \frac{\lambda_{k_1}}{\lambda_{k_1+j}} \frac{\alpha_{k_1,j}}{\alpha_{k_1,0}}. \tag{77}$$

However, since $\beta_0 \neq 0$, we also have

$$\frac{\lambda_n}{\lambda_{n+j}} = \frac{\lambda_0}{\lambda_j} \frac{\alpha_{0,j}}{\alpha_{0,0}}. \tag{78}$$

Comparing (77) and (78) we obtain

$$\lambda_{k_1+j} = \frac{\lambda_{k_1} \lambda_j}{\lambda_0} \frac{\alpha_{k_1 j}}{\alpha_{k_1 0}} \frac{\alpha_{00}}{\alpha_{0j}}. \tag{79}$$

Analogously, we show that

$$\lambda_{k_2+j} = \frac{\lambda_{k_2} \lambda_j}{\lambda_0} \frac{\alpha_{k_2 j}}{\alpha_{k_2 0}} \frac{\alpha_{00}}{\alpha_{0j}}. \tag{80}$$

From (79) or (80) we can write

$$\lambda_{nk+j} = \frac{\lambda_k^n \lambda_j}{\lambda_0^n} \left(\frac{\alpha_{00}}{\alpha_{k0}}\right)^n \frac{\alpha_{kj} \alpha_{k,j+k} \cdots \alpha_{k,j+(n-1)k}}{\alpha_{0j} \alpha_{0,j+k} \cdots \alpha_{0,j+(n-1)k}}. \tag{81}$$

If we now assume that $k_1 = nk_2 + s$, where $s < k_2$, we have

$$\begin{aligned}\lambda_{k_1+j} = \frac{\lambda_{k_2}^n \lambda_j \lambda_s}{\lambda_0^{n+1}} \frac{\alpha_{00}}{\alpha_{k_1 0}} \frac{\alpha_{k_1 j}}{\alpha_{0j}} \left(\frac{\alpha_{00}}{\alpha_{k_2 0}}\right)^n \\ \cdot \frac{\alpha_{k_2,s} \alpha_{k_2,s+k_1} \cdots \alpha_{k_2,s+(n-1)k_2}}{\alpha_{0,s} \alpha_{0,s+k_2} \cdots \alpha_{0,s+(n-1)k_2}}.\end{aligned} \tag{82}$$

Also

$$\begin{aligned}\lambda_{nk_2+s+j} = \frac{\lambda_{k_2}^n \lambda_{s+j}}{\lambda_0^n} \left(\frac{\alpha_{00}}{\alpha_{k_2 0}}\right)^n \\ \cdot \frac{\alpha_{k_2,s+j} \alpha_{k_2,s+j+k_2} \cdots \alpha_{k_2,s+j+(n-1)k_2}}{\alpha_{0,s+j} \alpha_{0,s+j+k_2} \cdots \alpha_{0,s+j+(n-1)k_2}}.\end{aligned} \tag{83}$$

Combining (82) and (83) we have

$$\lambda_{s+j} = \frac{\lambda_s \lambda_j}{\lambda_0} \phi(\alpha) \tag{84}$$

where

$$\begin{aligned}\phi(\alpha) = \left(\frac{\alpha_{00} \alpha_{k_1 j}}{\alpha_{k_1 0} \alpha_{0j}}\right) \frac{\alpha_{0,s+j} \cdots \alpha_{0,s+j+(n-1)k_2}}{\alpha_{k_2,s+j} \cdots \alpha_{k_2,s+j+(n-1)k_2}} \\ \cdot \frac{\alpha_{k_2,s} \alpha_{k_2,s+k_1} \cdots \alpha_{k_2,s+(n-1)k_2}}{\alpha_{0,s} \alpha_{0,s+k_2} \cdots \alpha_{0,s+(n-1)k_2}}.\end{aligned} \tag{85}$$

Note that $\phi(\alpha)$ is computable for every $j = 0, 1, 2, \cdots$. If $s = g$ we have a recursion similar to (75). Otherwise we define $\overline{k}_1 = k_2$ and $\overline{k}_2 = s$ and write $\overline{k}_1 = \overline{n}\overline{k}_2 + \overline{s}$. Finally, we repeat the computation described in (82) and (83) and rewrite a recursion for $\lambda_{\overline{s}+j}$. The procedure is continued until s is a factor of both k_1 and k_2. If $s = g$, we stop. Otherwise, we choose a k_3 such that $\beta_{k_s} \neq 0$.

Eventually since g is the g.c.f. of all $\{k: \beta_k \neq 0\} \cup \{n\}$, it follows that the algorithm will produce a recursion for λ_{g+j} of the form

$$\lambda_{g+j} = \frac{\lambda_g \lambda_j}{\lambda_0} \phi^*(\alpha) \tag{86}$$

where $\phi^*(\alpha)$ can be computed.

The following examples illustrate the use of the rescaling procedure described above.

Example 5.1: Consider the system given by (10) with

$$A=\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix},\quad x_0=\begin{pmatrix}3\\ -1\end{pmatrix},\quad C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{87}$$

The first five outputs of this system are given by

$$(z_0 \quad \cdots \quad z_4)=\begin{pmatrix}3 & -1 & 5 & 3 & 13\\ -1 & 5 & 3 & 13 & 19\end{pmatrix}. \tag{88}$$

Suppose that we observe these outputs scaled by

$$(\delta_0 \quad \cdots \quad \delta_4)=(1 \quad 2 \quad 3 \quad 1 \quad 2). \tag{89}$$

Thus, we observe $q_k=\delta_k z_k$ given by

$$(q_0 \quad \cdots \quad q_4)=\begin{pmatrix}3 & -2 & 15 & 3 & 26\\ -1 & 10 & 9 & 13 & 38\end{pmatrix}. \tag{90}$$

We now rescale the output using the algorithm just developed. We calculate the coefficients $\alpha_{k,j}, j=0,1,2, k=0,1$ to be

$$\begin{pmatrix}\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2}\\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2}\end{pmatrix}=\begin{pmatrix}6 & 1 & \frac{4}{3}\\ \frac{3}{2} & \frac{1}{3} & 2\end{pmatrix}. \tag{91}$$

Note that all the coefficients are nonzero, and thus we may use the formula given by (95). We choose $\lambda_0=\lambda_1=1$ and find the remaining rescale factors to be

$$\lambda_2=\tfrac{4}{3},\quad \lambda_3=8,\quad \lambda_4=8. \tag{92}$$

After rescaling the output as $w_k=\lambda_k q_k$, we have

$$(w_0 \quad \cdots \quad w_4)=\begin{pmatrix}3 & -2 & 20 & 24 & 208\\ -1 & 10 & 12 & 104 & 304\end{pmatrix}. \tag{93}$$

By comparing (93) with the original output (88), we see that the rescaling represents a scaling of the original output by

$$(\delta_0 \quad \cdots \quad \delta_4)=(1 \quad 2 \quad 4 \quad 8 \quad 16). \tag{94}$$

Since $\delta_1=2$ and $\delta_j=\delta_1^j$, it follows that $g=1$. By Lemma 3.8, such a scaling preserves the Hankel rank.

Finally, we also conclude from (14) that the set of all realizations perspectively equivalent to (87) and preserving $n=2, g=1, g_1=0$, is given up to changes of basis in the state space by

$$C=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1\end{pmatrix},\quad A=\begin{pmatrix}0 & \lambda_2\\ 2\lambda_2 & \lambda_2\end{pmatrix},\quad x_0=\begin{pmatrix}3\lambda_3\\ -\lambda_3\end{pmatrix} \tag{95}$$

where $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers. Alternatively from Section IV one can describe

$$A=\begin{pmatrix}0 & 1\\ 2\left(\dfrac{\delta_1}{\delta_0}\right)^2 & \dfrac{\delta_1}{\delta_0}\end{pmatrix},\quad x_0=\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$C=\begin{pmatrix}-3\delta_1 & 3\delta_0\\ 11\delta_1 & -\delta_0\end{pmatrix}. \tag{96}$$

For various values of δ_0 and δ_1, the triplets in (96) are in the space of controllable realizations of the perspective system (87) that preserves $n=2, g=1, g_1=0$.

Example 5.2: Consider the system in the previous example replacing A with

$$A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{97}$$

The first five outputs of this system are given by

$$(z_0 \quad \cdots \quad z_4)=\begin{pmatrix}3 & -1 & 3 & -1 & 3\\ -1 & 3 & -1 & 3 & -1\end{pmatrix}. \tag{98}$$

Once again, we observe the outputs scaled by

$$(\delta_0 \quad \cdots \quad \delta_4)=(1 \quad 2 \quad 3 \quad 1 \quad 2) \tag{99}$$

and get

$$(v_0 \quad \cdots \quad v_4)=\begin{pmatrix}3 & -2 & 9 & -1 & 6\\ -1 & 6 & -3 & 3 & -2\end{pmatrix}. \tag{100}$$

In this case, we calculate the coefficients $\alpha_{k,j}, j=0,1,2, k=0,1$ to be

$$\begin{pmatrix}\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2}\\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2}\end{pmatrix}=\begin{pmatrix}3 & \frac{1}{2} & \frac{2}{3}\\ 0 & 0 & 0\end{pmatrix}. \tag{101}$$

Now $\alpha_{1,j}=0\forall j$ so the formula given by (75) is not valid. We return to (73), and setting $k=0, j=1,2$, we derive the relationships

$$\frac{\lambda_2}{\lambda_3}=\left(\frac{\lambda_0}{\lambda_1}\right)\left(\frac{\alpha_{0,1}}{\alpha_{0,0}}\right) \tag{102}$$

and

$$\frac{\lambda_2}{\lambda_4}=\left(\frac{\lambda_0}{\lambda_2}\right)\left(\frac{\alpha_{0,2}}{\alpha_{0,0}}\right). \tag{103}$$

We choose $\lambda_0=\lambda_1=\lambda_2=1$ and find the remaining rescale factors to be

$$\lambda_3=6,\quad \lambda_4=\tfrac{9}{2}. \tag{104}$$

After rescaling the output as $w_k=\lambda_k v_k$, we have

$$(w_0 \quad \cdots \quad w_4)=\begin{pmatrix}3 & -2 & 9 & -6 & 27\\ -1 & 6 & -3 & 18 & -9\end{pmatrix}. \tag{105}$$

Once again we compare the rescaled output (105) with the original output (100). We find that the rescaling represents a scaling of the original output by

$$(\delta_0 \quad \cdots \quad \delta_4)=(1 \quad 2 \quad 3 \quad 6 \quad 9). \tag{106}$$

This sequence is of the form $\{1, \delta_1, \delta_2, \delta_1\delta_2, \delta_2^2, \cdots\}$ where $\delta_1=2, \delta_2=3$. Hence we have $g=2$.

The set of realizations of all perspective systems which preserve $n=2g=2, g_1=0$ is given by

$$C=\begin{pmatrix}-2\delta_1 & 3\delta_0\\ 6\delta_1 & -\delta_0\end{pmatrix},\quad A=\begin{pmatrix}0 & 1\\ \dfrac{\delta_2}{\delta_0} & 0\end{pmatrix},\quad x_0=\begin{pmatrix}0\\ 1\end{pmatrix} \tag{107}$$

up to changes in basis of the state space. Note in particular that (107) is described by three parameters.

We summarize the results of this section as follows. Given the perspectively observed output of a linear system, it is possible to rescale the output so as to preserve the Hankel rank of the original system and the invariants g_1 and g. In the nonsingular case, the formula for generating the rescaling constants is given by (75). Formulas for the singular cases can be derived, and the method is outlined in (86).

VI. Motion and Shape Estimation as a Perspective Identifiability Problem

Consider the following problem of motion and shape estimation, well known in machine vision.

Problem 6.1: A given planar textured surface is moving in discrete time following an affine recursion (109), where we assume that both the position of the surface and the parameters of the affine recursion are unknown. Assume that a camera produces a perfect image of the textured surface every instance of time. The problem of interest is to estimate the position and motion parameters of the surface from the observed time-varying image produced by the camera.

The above problem has been considered in [3], wherein the motion is assumed to be described in continuous time. In this paper we now obtain analogous results when the motion is assumed to be described in discrete time. For basic motivation, an introduction to the problem, and a literature survey, we would like to refer the reader to [3]. For a surface undergoing rigid motion and under perspective projection, the problem of motion and shape estimation has already been considered by Tsai and Huang in a series of papers [1], [12], [13], [14]. In fact, their approach can be regarded as discrete-time analogue of the treatment in continuous time due to Kanatani [15] and Waxman and Ullman [2].

We assume throughout this paper that we have a textured planar patch which faces a camera without any occlusion. Furthermore, we assume that every point on the surface moves according to a certain affine recursion. As a result of the motion of the individual points, the entire plane moves in time. In this section, we write down a recursion that describes the motion of the plane. We also specialize the equation to a planar patch undergoing a rigid motion.

Let us assume that (X, Y, Z) is the world coordinate frame, wherein we have a plane defined by the equation

$$Z = pX + qY + r. \tag{108}$$

We assume that p, q, r are functions of time. Furthermore, we assume that the motion field is given by the equation

$$\mathcal{X}_{k+1} = A\mathcal{X}_k + b \tag{109}$$

where A is an arbitrary 3×3 matrix and b is a 3×1 vector given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{110}$$

and

$$b = (b_1 \quad b_2 \quad b_3)^T \tag{111}$$

and where $\mathcal{X}$ is given by

$$\mathcal{X} = (X \quad Y \quad Z)^T.$$

We would now construct a difference equation that describes the motion of the shape parameters p, q, r. This is done as follows. Let us homogenize the vector $(X \quad Y \quad Z)$ as

$$X = \frac{\overline{X}}{\overline{W}}, \quad Y = \frac{\overline{Y}}{\overline{W}}, \quad Z = \frac{\overline{Z}}{\overline{W}}$$

and the vector $(p \quad q \quad r)$ as

$$p = \frac{\overline{p}}{\overline{s}}, \quad q = \frac{\overline{q}}{\overline{s}}, \quad r = \frac{\overline{r}}{\overline{s}}. \tag{112}$$

If we denote

$$\begin{aligned} \mathcal{P}_k^T &= (\overline{p}_k, \overline{q}_k, -\overline{s}_k, \overline{r}_k) \\ \overline{\mathcal{X}}_k^T &= (\overline{X}_k, \overline{Y}_k, \overline{Z}_k, \overline{W}_k) \end{aligned}$$

it follows that one can rewrite (108) as

$$\mathcal{P}_k^T \overline{\mathcal{X}}_k = 0 \tag{113}$$

and (109) as

$$\overline{\mathcal{X}}_{k+1} = \mathcal{A}\overline{\mathcal{X}}_k \tag{114}$$

where

$$\mathcal{A} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}. \tag{115}$$

From (113) and (114) it follows that

$$\mathcal{P}_k = \frac{1}{\alpha} A^T \mathcal{P}_{k+1} \tag{116}$$

where α is any nonzero scalar. Equation (116) describes how the shape parameters change in time and is therefore called the shape dynamics. Note that the shape dynamics (116) evolves backward in time and describes the motion of the plane (108) as points of the plane follow the recursion (109).

Note also that (116) is parameterized by 12 motion parameters and initial conditions on three shape parameters. Thus there is a total of 15 parameters describing the shape dynamics (116) for the affine motion.

An important special case of the affine motion (109) is the case when $A = e^{\Omega}$, where Ω is a skew symmetric matrix given by

$$\begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix} \triangleq \Omega. \tag{117}$$

Under this assumption, the motion field (109) describes a rigid motion. Equation (116) is parameterized by a total of six motion parameters and initial conditions on three shape parameters. Thus there is a total of nine parameters describing the shape dynamics (116) for the rigid motion. We also note that in the case when $A = e^{\Omega}$, the matrix A is nonsingular and therefore the shape dynamics can also be propagated forward in time.

Assume that the surface described by (108) is textured, i.e., the intensity $E(X, Y, Z, k)$ of a point (X, Y, Z) on the surface at time k does not change along the solution of (109). We also assume that the camera is perfectly focused on the object surface, i.e., intensity from a surface on the object to the image plane is transferred unattenuated under the camera correspondence. The above two assumptions together imply that the intensity of a point along the solution of (109) does not change as the point is projected on the image plane. In this section we consider the projection to be described as follows.

Let (x, y) be the coordinates of the image plane obtained under the projection of a point (X, Y, Z) on the surface of the object. We define

$$x = \frac{fX}{Z+\delta}, \quad y = \frac{fY}{Z+\delta} \tag{118}$$

where $\delta \in [0, f]$ and f is the focal length of the camera. Note that if $\delta = 0$ we obtain a viewer-centered projection. If $\delta = f$, we obtain an image centered projection. These two projections have been described in [15]. Finally note that if $\delta = f$ and $f \to \infty$, we obtain

$$x = X, \quad y = Y \tag{119}$$

which is known in the literature [15] as the "orthographic projection."

We assume that every point of the plane (108) is projected via generalized projection (118). The "optical flow" equation, described as a recursive equation satisfied by the coordinates (x_k, y_k) of the projected point, can be written as follows:

$$\begin{aligned} x_{k+1} &= \frac{d_1 x_k + d_2 y_k + f d_3}{\frac{1}{f} d_7 x_k + \frac{1}{f} d_8 y_k + d_9} \\ y_{k+1} &= \frac{d_4 x_k + d_5 y_k + f d_6}{\frac{1}{f} d_7 x_k + \frac{1}{f} d_8 y_k + d_9} \end{aligned} \tag{120}$$

where

$$\begin{aligned} d_1 &= a_{11} - c_1 p_k \\ d_2 &= a_{12} - c_1 q_k \\ d_3 &= a_{13} + c_1 \\ d_4 &= a_{21} - c_2 p_k \\ d_5 &= a_{22} - c_2 q_k \\ d_6 &= a_{23} + c_2 \\ d_7 &= a_{31} - c_3 p_k \\ d_8 &= a_{32} - c_3 q_k \\ d_9 &= a_{33} + c_3 \end{aligned} \tag{121}$$

and where

$$\begin{aligned} c_1 &= \frac{b_1 - a_{13}\delta}{r+\delta} \\ c_2 &= \frac{b_2 - a_{23}\delta}{r+\delta} \\ c_3 &= \frac{b_3 - (a_{33}-1)\delta}{r+\delta}. \end{aligned} \tag{122}$$

The pair (116), (121) can be viewed as a perspective dynamical system in discrete time described by

$$\mathcal{P}_k = \mathcal{A}^T \mathcal{P}_{k+1} \tag{123}$$

$$\xi_k = \begin{pmatrix} -b_1' & 0 & -a_{11}\delta & a_{11} \\ 0 & -b_1' & -a_{12}\delta & a_{12} \\ 0 & 0 & -b_1 & a_{13} \\ -b_2' & 0 & -a_{21}\delta & a_{21} \\ 0 & -b_2' & -a_{22}\delta & a_{22} \\ 0 & 0 & -b_2 & a_{23} \\ -b_3' & 0 & -a_{31}\delta & a_{31} \\ 0 & -b_3' & -a_{32}\delta & a_{32} \\ 0 & 0 & -b_3 - \delta & a_{33} \end{pmatrix} \mathcal{P}_k \tag{124}$$

where $[\mathcal{P}_k] \in \mathbb{RP}^3$ and $[\xi_k] \in \mathbb{RP}^8$ and

$$\begin{aligned} b_1' &= b_1 - a_{13}\delta \\ b_2' &= b_2 - a_{23}\delta \\ b_3' &= b_3 - (a_{33} - 1)\delta \end{aligned} \tag{125}$$

and where

$$[\xi_k] = [d_1 d_2 \cdots d_9]$$

in homogeneous coordinates. To summarize this section, if a planar surface undergoes affine motion, one can view the motion of the shape parameters as a dynamical system and the coefficients of the optical flow as the observation function. This gives rise to a perspective system (123), (124) that has the same form as introduced in (10), (11). Note that motion parameters A, b parameterize the dynamics of the perspective system (123), (124).

In the next section we show that the vector $[d_1 \cdots d_9]$ (which we call the essential parameter vector) can be computed by observations of various features on the image plane. Once computed, it will be treated as the observation vector for (123). Finally, our goal in Section VIII is to study the parameter identification problem for (123), (124) as introduced in Section III. In Section VIII we shall also study as a special case the parameter identification of the perspective system (123), (124) when $f = \delta$ and $f \to \infty$. This way we recover the case when the camera views feature points via "orthographic projection."

VII. Estimation of the Essential Parameters

In this section, we shall consider the problem of estimating the essential parameter vector $(d_1, \cdots, d_9)$ up to a nonzero scale factor. Let $e(x, y, k)$ be the intensity function on the image plane produced by the moving plane (108). We shall assume that the intensity function remains invariant under the dynamics of the projected point given by (120). Let us define

$$\begin{aligned} \phi(x,y) &= \frac{d_1 x + d_2 y + f d_3}{\frac{1}{f} d_7 x + \frac{1}{f} d_8 y + d_9} \\ \psi(x,y) &= \frac{d_4 x + d_5 y + f d_6}{\frac{1}{f} d_7 x + \frac{1}{f} d_8 y + d_9}. \end{aligned} \tag{126}$$

We shall now describe two different methods of computing the vector $(d_1, \cdots, d_9)$.

A. Computation Based on Optical Flow

We assume that the intensity does not change along the solution of the optical flow (120), i.e.,

$$e(x, y, k) = e(\phi(x,y), \psi(x,y), k+1). \tag{127}$$

Equation (127) is an implicit description of the intensity dynamics on the image plane and is analogous to the intensity dynamics derived in [3]. Up to first-order terms one can approximate (127) as follows:

$$e(x,y,k) = e(x,y,k+1) + \frac{\partial e(x,y,k+1)}{\partial x}(\phi(x,y)-x) + \frac{\partial e(x,y,k+1)}{\partial y}(\psi(x,y)-y). \tag{128}$$

Equation (128) describes the intensity dynamics described backward in time. From (126) and (128) we have

$$\begin{aligned}(d_7x + d_8y + fd_9)&(e(x,y,k) - e(x,y,k+1)) \\ &= (e_x \quad e_y) \cdot \begin{bmatrix} f(d_1x + d_2y + d_3f) - x(d_7x + d_8y + d_9f) \\ f(d_4x + d_5y + d_6f) - y(d_7x + d_8y + d_9f) \end{bmatrix}\end{aligned} \tag{129}$$

where

$$e_x \triangleq \frac{\partial e(x,y,k+1)}{\partial x} \quad e_y = \frac{\partial e(x,y,k+1)}{\partial y}.$$

The parameters $(d_1,\cdots,d_9)$ can now be computed up to a nonzero scale factor by solving linear equations (129) from data obtained from sufficiently many points on the screen.

B. Computation Based on Dynamics of a Discontinuous Curve

If we assume that at the time instance k, the intensity function is discontinuous along the curve

$$y = I_k(x) \tag{130}$$

on the image plane, the dynamics of the discontinuity curve (130) can be used to compute the essential parameter vector up to a nonzero scale factor. In fact from (120), (126), and (130) we have

$$\psi(x,y) = I_{k+1}(\phi(x,y)). \tag{131}$$

Up to first-order terms, we approximate (131) as

$$\psi(x,y) = I_{k+1}(x) + \frac{\partial I_{k+1}}{\partial x}(\phi(x,y) - x). \tag{132}$$

From (126) and (132) we obtain

$$(d_1 \quad d_2 \quad \cdots \quad d_9) \cdot \begin{pmatrix} x\frac{I_{k+1}}{\partial x} \\ I_k\frac{\partial I_{k+1}}{\partial x} \\ f\frac{\partial I_{k+1}}{\partial x} \\ -x \\ -I_k \\ -f \\ -\frac{x^2}{f}\frac{\partial I_{k+1}}{\partial x} + \frac{x}{f}I_{k+1}(x) \\ -\frac{1}{f}xI_k(x)\frac{\partial I_{k+1}}{\partial x} + \frac{1}{f}I_k(x)I_{k+1}(x) \\ -x\frac{\partial I_{k+1}}{\partial x} + I_{k+1}(x) \end{pmatrix} = 0. \tag{133}$$

Equation (133) can be solved for the vector $(d_1,\cdots,d_9)$ up to a nonzero scale factor, provided that the data has been obtained from sufficiently many points in the image plane.

VIII. Identifiability Results for Motion and Shape Estimation

In this section we shall consider the problem of parameter identification for the perspective system (123), (124). Recall from Theorem 1.7 that generically parameters can be identified up to action of the perspective group $\mathcal{G}$ described in (13) and (14). In fact, the generic set is precisely characterized by Assumption 3.1 and the fact that the integers n, g, g_1 are to be preserved. Additionally, since the matrices in the system (123), (124) have a special structure, one considers a subgroup of $\mathcal{G}$ that preserves this structure.

Our analysis is based on considering three subcases of Problem 1.6. These subcases are affine motion under perspective projection (118), affine motion under orthographic projection (119), and rigid body motion under orthographic projection (120). In each case, we present a theorem describing the extent to which motion and shape parameters can be identified. In the case of perspective projection (118), one cannot improve on the result obtained for affine motion, thus the case of rigid motion is not considered separately.

A. Affine Motion Under Perspective Projection

The most general case that we shall consider is that of affine motion (109) under perspective projection (118). We show that one can identify the motion and shape parameters up to a one-parameter family. This result is quite surprising since there are 15 motion and shape parameters and only eight independent output functions given by the coefficients of the optical flow. Thus by including the shape dynamics in our analysis, we obtain much stronger results than if we had only considered the optical flow (as has been the case in [15] and [2]). We have the following theorem.

Theorem 8.1: Consider the perspective dynamical system given by (123) and (124). Suppose that the coefficients of the characteristic polynomial of $\mathcal{A}$ are all nonzero. Suppose furthermore that the following genericity conditions hold:

$$\begin{aligned} b_1'a_{23} - b_2'a_{13} &\neq 0 \\ b_2'a_{31} - b_3'a_{21} &\neq 0 \\ b_1'a_{32} - b_3'a_{12} &\neq 0 \end{aligned} \tag{134}$$

$$b'^T \neq 0 \tag{135}$$

$$\begin{pmatrix} A - I \\ b'^T \end{pmatrix} \quad \text{has rank 3} \tag{136}$$

where

$$b'^T = (b_1' \quad b_2' \quad b_3') \tag{137}$$

and b_1', b_2', and b_3' are given by (125). If the set of vectors $\{\xi_0, \xi_1, \cdots, \xi_{n-1}\}$ are linearly independent, then the parameters that can be locally identified via perspective observation are

$$(A, p, q, c_1, c_2, c_3) \tag{138}$$

where c_1, c_2, c_3 are defined as in (122).

Remark 8.2: We note that the condition in Theorem 8.1 that the coefficients of the characteristic polynomial of $\mathcal{A}$ are all nonzero ensures that the g.c.f. g in Lemma 3.8 is 1. Thus by Lemma 3.8, an output generated by a perspective system other than (123), (124) differs from ξ_k by only a scaling of the form $1, \delta, \delta^2, \delta^3, \cdots$. The scaling generated by the perspective group $\mathcal{G}$ is precisely such a scaling.

Proof of Theorem 8.1: It follows from Remark 8.2 and Lemma 3.8 that the parameters of the system (123), (124) lie in the orbit of the group $\mathcal{G}$ given by (13) and (14) acting on $(\Delta, \mathcal{A}, \mathcal{P}_0)$. We now restrict the action of $\mathcal{G}$ to the case where λ_2 in (14) has been chosen to be 1. In fact, if there are other real nonunity eigenvalues of A, there are other choices of λ_2 equal to the number of such eigenvalues (see [9] for details). In fact, if λ is such an eigenvalue one can choose $\lambda_2 = 1/\lambda$. Hence the parameter set (138) is ambiguous up to such finitely many choices of the group element λ_2.

Let Δ denote the 9×4 matrix in (124). In general, this group action changes the structure of the matrix Δ and $\mathcal{A}$. We now determine the subgroup of $GL(4)$ which preserves the structure. Clearly, parameters are identifiable up to orbits of this subgroup. We show that under the assumption of genericity (134)–(136), the only subgroup of $GL(4)$ which preserves the structure of $(\Delta, \mathcal{A})$ under the action (13) is given by

$$\overline{P} = \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{11} & 0 \\ 0 & 0 & \delta\alpha_{11} & \alpha_{44} \end{pmatrix} \tag{139}$$

where $\alpha_{11} \neq 0, \alpha_{44} \neq 0$. Let

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \delta & 1 \end{pmatrix}. \tag{140}$$

It is easy to see that

$$\mathcal{A}_1 \triangleq Q^{-1}\mathcal{A}Q = \begin{pmatrix} -A^T & 0 \\ -b'^T & 1 \end{pmatrix} \tag{141}$$

$$\Delta_1 \triangleq \Delta Q = \begin{pmatrix} -b'_1 & 0 & 0 & a_{11} \\ 0 & -b'_1 & 0 & a_{12} \\ 0 & 0 & -b'_1 & a_{13} \\ -b'_2 & 0 & 0 & a_{21} \\ 0 & -b'_2 & 0 & a_{22} \\ 0 & 0 & -b'_2 & a_{23} \\ -b'_3 & 0 & 0 & a_{31} \\ 0 & -b'_3 & 0 & a_{32} \\ 0 & 0 & -b'_3 & a_{33} \end{pmatrix}. \tag{142}$$

Let $Q_1 = (\alpha_{ij})$ be a nonsingular 4×4 matrix, i.e., an element of $GL(4)$. By the genericity condition (134), it may be concluded that $\Delta_1 Q_1$ has the same structure as Δ_1 if Q_1 has the form

$$Q_1 = \begin{pmatrix} \alpha_{11} I & \Theta \\ 0 & \alpha_{44} \end{pmatrix} \tag{143}$$

where

$$\Theta = \begin{pmatrix} \alpha_{14} \\ \alpha_{24} \\ \alpha_{34} \end{pmatrix}. \tag{144}$$

To ensure that $Q_1^{-1}\mathcal{A}_1 Q_1$ has the same structure as $\mathcal{A}_1$, we must have

$$b'^T\Theta = 0 \quad \text{and} \quad A^T\Theta = 0.$$

By genericity conditions (135) and (136), it follows that $\Theta = 0$. Thus Q_1 must be of the form

$$Q_1 = \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{11} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix} \tag{145}$$

to ensure the special structure of Δ_1 and $\mathcal{A}_1$.

The structure (139) of the $\overline{P}$ matrix is obtained by defining $\overline{P} = QQ_1$, where Q, Q_1 are given by (140) and (145), respectively. Thus we have shown that $\overline{P}$ defines the subgroup of $GL(4)$ that preserves the structure of Δ and $\mathcal{A}$. It is easy to see that the function $b'/(r + \delta)$ is invariant under this subgroup action. Thus the parameters (138) remain invariant under the subgroup action. Considering Remark 8.2, we may conclude that no two orbits of the $\mathcal{G}$ action under the restriction that $\lambda_2 = 1$ in (138) produce the same output $[\xi]$. Thus the functions of (138) are locally identifiable. ■

Remark 8.3: By local identification, we mean that in a sufficiently small neighborhood of the parameters, (A, p, q, c_1, c_2, c_3), there is no other parameter that would produce the same output sequence $[\xi_j], j = 0, 1, 2, \cdots$ described by (124). In fact if λ is any real eigenvalue of $A, \lambda \neq 1$, it can be shown [9] that there is another set of parameters $(A^*, p^*, q^*, c_1^*, c_2^*, c_3^*)$ distinct from the above set which would produce the same output sequence (124). A^* can be written as follows:

$$A^{*T} = \frac{1}{\lambda}A^T + \left(1 - \frac{1}{\lambda}\right)\frac{\theta b'^T}{b'^T\theta}$$

where θ is the eigenvector of A corresponding to eigenvalue λ and b' is defined in (137). The parameters $p^*, q^*, c_1^*, c_2^*, c_3^*$ can be described likewise, but the method has been omitted.

Remark 8.4: We note that by Theorem 8.1, it is possible to identify (locally) the motion and shape parameters up to the single scale factor expressed by the ambiguity of b_i and r in the definition of c_i (122). Furthermore, it is clear that this ambiguity is fundamental under perspective projection and, as such, is not removed by restricting the motion to that of a rigid body. Thus one cannot improve Theorem 8.1 under such a restriction. We have rigid body motion if in (115) we replace the matrix A with e^{Ω}, where Ω is defined as in (108). Such a matrix A would not have a real eigenvalue other than 1. The parameters that can be identified globally in this case are

$$(\omega_1, \omega_2, \omega_3, p, q, c_1, c_2, c_3). \tag{146}$$

Thus the only improvement of Theorem 8.1 under the restriction of a rigid motion is that the local identification is in fact global.

B. Affine Motion Under Orthographic Projection

Orthographic projection is obtained as a special case when the projection (118) degenerates to (119). This happens precisely when we assume $f = \delta$ and take the limit when $f \to \infty$. In a physical situation, objects focused far away compared to its size are best modeled by orthographic projection. Under the above limit, (120) takes up the form

$$\begin{aligned} x_{k+1} &= \frac{d_1 x_k + d_2 y_k + d_3}{d_9} \\ y_{k+1} &= \frac{d_4 x_k + d_5 y_k + d_6}{d_9} \end{aligned} \tag{147}$$

where

$$\begin{aligned} d_1 &= a_{11} + a_{13} p_k \\ d_2 &= a_{12} + a_{13} q_k \\ d_3 &= a_{13} r + b_1 \\ d_4 &= a_{21} + a_{23} p_k \\ d_5 &= a_{22} + a_{23} q_k \\ d_6 &= a_{23} r + b_2 \\ d_9 &= 1. \end{aligned}$$

The parameters d_7 and d_8 in (120) are not observed. The coefficients of the optical flow (147) give rise to the following homogeneous output vector:

$$[\xi_k] = [d_3 \quad d_6 \quad d_1 \quad d_2 \quad d_4 \quad d_5 \quad d_9]. \tag{148}$$

The perspective dynamical system (123)–(125) degenerates to

$$\mathcal{P}_k = A^T \mathcal{P}_{k+1} \tag{149}$$

$$\xi_k = \begin{pmatrix} a_{13} & 0 & -a_{11} & 0 \\ 0 & a_{13} & -a_{12} & 0 \\ 0 & 0 & -b_1 & a_{13} \\ a_{23} & 0 & -a_{21} & 0 \\ 0 & a_{23} & -a_{21} & 0 \\ 0 & 0 & -b_2 & a_{23} \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathcal{P}_k. \tag{150}$$

The perspective system (149), (150) has already been studied in detail in [3]. Analogous to the proof of Theorem 8.1, it can be shown that there is a subgroup of the general linear group $GL(4)$ of the form

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 \\ 0 & \alpha_{11} & \alpha_{23} & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{11} \end{pmatrix} \tag{151}$$

where $\alpha_{11} \neq 0, \alpha_{33} \neq 0$ such that (13) replaced by the subgroup (151) would preserve the structure of the system (149), (150). If we define

$$\begin{aligned} \pi_1 &= \alpha_{11}/\alpha_{33} \\ \pi_2 &= \alpha_{13}/\alpha_{33} \\ \pi_3 &= \alpha_{23}/\alpha_{33} \\ \pi_4 &= \alpha_{43}/\alpha_{33} \end{aligned}$$

the orbit of the group (151) in the parameter space is described by

$$\begin{aligned} a_{11} &\mapsto a_{11} - \pi_2 a_{13} \\ a_{21} &\mapsto a_{21} - \pi_2 a_{23} \\ a_{12} &\mapsto a_{12} - \pi_3 a_{13} \\ a_{22} &\mapsto a_{22} - \pi_3 a_{23} \\ a_{13} &\mapsto \pi_1 a_{13} \\ a_{23} &\mapsto \pi_1 a_{23} \\ b_1 &\mapsto b_1 - \pi_4 a_{13} \\ b_2 &\mapsto b_2 - \pi_4 a_{23} \\ a_{31} &\mapsto \frac{1}{\pi_1}(\pi_2 a_{11} + \pi_3 a_{21} + a_{31}) \\ &\quad - \frac{\pi_2}{\pi_1}(\pi_2 a_{13} + \pi_3 a_{23} + a_{33}) \\ a_{32} &\mapsto \frac{1}{\pi_1}(\pi_2 a_{12} + \pi_3 a_{22} + a_{32}) \\ &\quad - \frac{\pi_3}{\pi_1}(\pi_2 a_{13} + \pi_3 a_{23} + a_{33}) \\ b_3 &\mapsto \frac{1}{\pi_1}(\pi_2 b_1 + \pi_3 b_2 + b_3) \\ &\quad - \frac{\pi_4}{\pi_1}(\pi_2 a_{13} + \pi_3 a_{23} + a_{33}) \\ a_{33} &\mapsto \pi_2 a_{13} + \pi_3 a_{23} + a_{33} \\ p &\mapsto \frac{1}{\pi_1}(p + \pi_2) \\ q &\mapsto \frac{1}{\pi_1}(q + \pi_3) \\ r &\mapsto \frac{1}{\pi_1}(r + \pi_4). \end{aligned} \tag{152}$$

Equation (152) describes a four-parameter orbit in the parameter space. The following theorem shows that generically the orbits are identifiable.

Theorem 8.5: Consider the perspective dynamical system given by (149) and (150). Suppose that the coefficients of the characteristic polynomial of $\mathcal{A}$ are all nonzero. Suppose further that the following genericity conditions hold:

$$\begin{aligned} a_{13} &\neq 0 \\ b_1 a_{23} - b_2 a_{13} &\neq 0 \\ a_{12} a_{23} - a_{13} a_{22} &\neq 0 \\ a_{11} a_{23} - a_{13} a_{21} &\neq 0. \end{aligned} \tag{153}$$

Then it follows that parameters can be identified up to a four-parameter orbit of a subgroup of $GL(4)$ described by (151), the orbit being described by (152).

The proof of Theorem 8.5 is analogous to that of Theorem 8.1 and is exactly the same as in the continuous time case. The reader is referred to [3] for details.

C. Rigid Body Motion Under Orthographic Projection

Finally, we consider the case where the motion (109) is restricted to that of a rigid body. We replace the matrix A by e^{Ω}, where Ω is given by (117). In this case we have nine motion and shape parameters and only six independent coefficients of optical flow. By exploiting the special structure of the A matrix, we find that one can identify the motion and shape parameters up to a one parameter family and a sign ambiguity.

Theorem 8.6: Consider the perspective dynamical system given by (149) and (150) with the restriction that $A = e^{\Omega}, \Omega$ given by (117). Suppose that the coefficients of the characteristic polynomial of $\mathcal{A}$ are all nonzero and that the system satisfies the genericity condition (153). Then the parameters that can be identified up to a sign ambiguity are

$$\omega_1, \pm\omega_2, \pm\omega_3, b_1 \mp \frac{\alpha_{43}}{\alpha_{11}}\omega_2, b_2 \mp \frac{\alpha_{43}}{\alpha_{11}}\omega_3, \pm b_3, \pm p, \pm q, \pm r + \frac{\alpha_{43}}{\alpha_{11}}. \tag{154}$$

Proof: The additional constraint imposed by rigid body motion is that the matrix A be orthonormal. That is

$$A^T A = I. \tag{155}$$

Let us consider the orthonormality constraint in more detail for the first two rows of A. We have

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1 \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0. \end{aligned} \tag{156}$$

The expressions listed in (152) define a four-dimensional orbit in the 15-dimension parameter space. We may express the constraints given by (156) in this orbit as

$$\begin{bmatrix} -2a_{11} & -2a_{12} & a_{13} \\ -2a_{21} & -2a_{22} & a_{23} \\ -a_{13}a_{21} - a_{11}a_{23} & -a_{13}a_{22} - a_{12}a_{23} & a_{13}a_{23} \end{bmatrix} \cdot \begin{bmatrix} \pi_2 \\ \pi_3 \\ \gamma \end{bmatrix} = 0 \tag{157}$$

where

$$\gamma = \pi_1^2 + \pi_2^2 + \pi_3^2 - 1 \tag{158}$$

and where π_1, π_2, and π_3 are free variables in the orbit. By our genericity assumption, the 3×3 matrix in (157) is nonsingular. Thus we must have $\pi_2 = \pi_3 = \gamma = 0$. And so, by (158) we have $\pi_1 = \pm 1$.

We must check that under this restriction of π_1, π_2, and π_3, the orthonormality condition on A still holds. The orbit of the A matrix has been restricted to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & \pm a_{13} \\ a_{21} & a_{22} & \pm a_{23} \\ \pm a_{31} & \pm a_{32} & a_{33} \end{bmatrix}. \tag{159}$$

Thus by inspection we see that the orthonormality condition is preserved. Finally, the parameters that can be recovered follow from (152) by requiring that $\pi_2 = \pi_3 = 0$ and $\pi_1 = \pm 1$. ■

We conclude that the rigid body motion reduces the orbit of identifiable parameters from a four-parameter orbit to two copies of a one-parameter orbit. This result corresponds to the result in the continuous-time case (see [3]).

IX. Conclusion

In this paper we analyze observability and identifiability conditions for a perspective system and describe a new rescaling algorithm for perspective systems identification. The observability condition generalizes the well-known Hautus' condition on the observability of linear dynamical systems (see [7] and [8]). The identifiability condition also generalizes concepts well known in linear dynamical systems via parameters that can be identified up to orbits of a group, provided they meet a certain genericity restriction and provided the dimension p of the output vector is sufficiently large ($p \geq n - g_1/g$ would suffice). In this paper we show that generically parameters of a perspective dynamical system can be recovered up to orbits of a perspective group. The structure of the perspective group, however, depends on n, the dimension of the state space and a pair of integers "g" and "g_1" which can be computed from the parameters of the system. Furthermore, we show that generically $g = 1$ and $g_1 = 0$. The results on parameter identification are applied to motion and shape estimation problems of a planar surface undergoing affine or rigid motion.

As a final remark we would like to point out that in this paper we have approached the problem of perspective system identification by posing and analyzing the perspective system realization problem. The analysis has been carried out for a discrete-time system as opposed to that of [3], where this was done for a continuous-time system. We note in passing that discrete-time realization theory of perspective systems appears to be more complicated than its continuous-time counterpart because of the role played by the pair of integers g and g_1. These do not have a corresponding analogue in continuous time.

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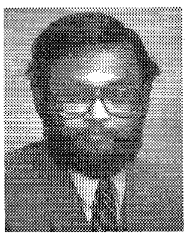

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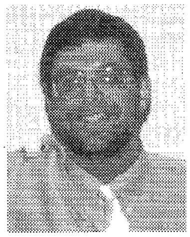

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