Technical Notes and Correspondence

On Undershoot and Nonminimum Phase Zeros

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Abstract—**In this note we derive a simple necessary and sufficient condition for a stable system to exhibit an undershooting step response. Specifically, we show that undershoot occurs if and only if the plant has an odd number of real right-half plane zeros.**

Consider a lumped scalar system with a strictly proper transfer function $p(s)$, and suppose the system is stable. Let $y(\cdot)$ denote the step response of the system. Then by stability, the limit $y(\infty)$ is well-defined and equals $p(0)$. Let r denote the relative degree of p. Then y and its first $r-1$ derivatives are zero at $t = 0$, and $y^{(r)}(0)$ is the first nonzero derivative. If $r = 1$ then the step response is continuous at $t = 0$ but $\dot{y}(0) \neq 0$. Within the control community, a folklore definition is that the step response exhibits "undershoot" if it "initially starts off in the wrong direction." However, to date no precise definition is available. For the purposes of this note, we adopt the definition that the step response exhibits undershoot if its steady-state value has a sign opposite from that of its first nonzero derivative at time $t = 0$. Thus, we define a system to *have undershoot* if $y^{(r)}(0)y(\infty) < 0$. Clearly, this definition only makes sense if $p(0) = y(\infty) \neq 0$. This is a natural mathematical version of "the step response initially starts in the wrong direction." Then we have the following result, which is very easy to prove but does not seem to appear anywhere.

Proposition: The system has undershoot if and only if its transfer function has an odd number of real RHP zeros.

Proof: We can assume that $p(0) = 1$ without loss of generality, since the presence or absence of undershoot is not affected by dividing $p(s)$ by a nonzero constant. As for $y^{(r)}(0)$, the initial value theorem tells us that

$$y^{(r)}(0) = \lim_{s\to\infty} s^r p(s). \tag{1}$$

Now write $p(s)$ in the form

$$p(s) = \frac{\prod_{i=1}^{n}\left[1-\frac{s}{z_i}\right]}{\prod_{i=1}^{n+r}\left[1-\frac{s}{p_i}\right]}. \tag{2}$$

The numerator terms can be grouped into three types: i) those corresponding to positive real zeros, ii) those corresponding to negative real zeros, and iii) those corresponding to complex zeros. Now the first are of the form

$$1-\alpha_i s. \tag{3}$$

For some positive α_i, while the second are of the form

$$1+\alpha_i s \tag{4}$$

for some positive α_i. The third terms are of the form

$$1+\beta_i s+\alpha_i s^2, \tag{5}$$

for some positive α_i, although of course β_i could be negative. Also, we must have $\beta_i^2 < 4\alpha_i$ to ensure that the corresponding zero is complex. On the other hand, by the assumption that p is stable, all terms in the denominator are of the form (4) or (5). Since

$$y^r(0) = \lim_{s\to\infty} s^r p(s), \tag{6}$$

we see that the sign of $y^{(r)}(0)$ is determined solely by the number of terms of the type (3). Specifically, if their number is odd, then $y^{(r)}(0)$ is negative, whereas if their number is even, then $y^{(r)}(0)$ is positive. Since $y(\infty) = p(0) = 1$, we have the desired conclusion.

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Simultaneous Partial Pole Placement: A New Approach to Multimode System Design

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Abstract—**Simultaneous partial pole placement of a family of single-input single-output plants is proposed as a generalization of the classical pole placement and stabilization problems. This problem finds application in the design of a compensator for a family of linear dynamical systems. In this note we show that the proposed problem is equivalent to a new class of transcendental problem using stable, minimum phase rational functions with real coefficients. A necessary condition for the solvability of the associated transcendental problem is obtained. Finally, a counterexample to the following conjecture is obtained—"pairs of simultaneously stabilizable plants of bounded McMillan degree have simultaneously stabilizing compensators of bounded McMillan degree."**

NOTATIONS

$\mathfrak{C}$	The complex plane.
$\mathfrak{C}_s$	A self conjugate open subset of the complex plane which intersects the real axis $\mathfrak{R}$.
$\mathfrak{C}_u$	$[\mathfrak{C} \cup \{\infty\}] - \mathfrak{C}_s$.
C^-	Open left half of the complex plane.
$\mathfrak{C}^+$	Closed right half of the complex plane including infinity.
ID^-	Open interior of the unit disk.
ID^+	Closed exterior of the unit disk including infinity.
H	Ring of proper rational functions with real coefficients with poles in $\mathfrak{C}_s$.
J	Set of multiplicative units in H.
Λ	A parameter set.

I. STATEMENT OF THE SIMULTANEOUS PARTIAL POLE PLACEMENT PROBLEM

The simultaneous partial pole placement problem consists of answering the following question.

"Given a family $g_\lambda(s)$ of single-input single-output proper transfer functions of degree n_λ, $\lambda \in \Lambda$, respectively. Given a family of nonzero biproper (i.e., proper but not strictly proper) transfer functions $\psi_\lambda(s)$ of degree d_λ, $\lambda \in \Lambda$, respectively, with poles in a suitable open subset $\mathfrak{C}_s$ of the complex plane $\mathfrak{C}$ and zeros in $\mathfrak{C}_u$. Does there exist a proper compensator $k(s)$ of degree $q \geq \max\ [d_\lambda - n_\lambda]$ such that the closed-loop systems $g_\lambda(s)[1 + k(s)g_\lambda(s)]^{-1}$, $\lambda \in \Lambda$ have, respectively, d_λ poles in $\mathfrak{C}_u$ where $\psi_\lambda(s)$ vanishes and all but the above d_λ poles are in $\mathfrak{C}_s$ for $\lambda \in \Lambda$, respectively?"

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II. Some Remarks and Motivation

Classical choices of the region $\mathfrak{S}_s$ are $\mathfrak{S}^-$ and ID^- for continuous-time and discrete-time stabilization problems, respectively. A different choice of $\mathfrak{S}_s$ is motivated by the following example where we assume Λ to be a single point, i.e., where the family $g_\lambda(s)$ consists of only one plant.

Example 2.1 (The Dominant Pole Placement Problem): "Let σ_1, σ, $\omega \in \mathfrak{R}$ and assume $\sigma_1 \gg \sigma > 0$. Let $g(s)$ be a plant (in continuous time) of McMillan degree n. The problem is to design a compensator of degree $\geq \max(2-n, 0)$ such that two of the closed-loop poles are at $-\sigma \pm j\omega$, while all the other closed-loop poles are in the region $\mathfrak{S}_s \triangleq \{s: \text{Re } s < -\sigma_1\}$."

The above problem is known as the dominant pole placement problem since the impulse response of the closed-loop system is dominated by the response due to the pair of poles at $-\sigma \pm j\omega$. Let $\alpha_1, \alpha_2 \in \mathfrak{R}$ and assume $\alpha_1, \alpha_2 > \sigma_1$. It follows by choosing

$$\psi(s) = [(s+\sigma)^2 + \omega^2]/[(s+\alpha_1)(s+\alpha_2)]$$

that the dominant pole placement problem is a special case of the proposed problem where Λ is a single point.

Remark 2.2: Assume Λ to be the finite set $\{1, \cdots, r\}$, i.e., the family $g_\lambda(s)$ consists of a collection of plants $g_1(s), \cdots, g_r(s)$. If $d_1 = d_2 = \cdots = d_r = 0$, i.e., if $\psi_i(s)$, $i = 1, \cdots, r$ are all nonzero constants, the proposed partial pole placement problem specializes to the simultaneous stabilization problem studied in [1], [4], [6], and [9]. On the other hand, for a fixed q, if $d_i = n_i + q$, $i = 1, \cdots, r$; one obtains the simultaneous pole placement problem [2], wherein the compensator $k(s)$ places the poles of the ith plant $g_i(s)$ at the zeros of $\psi_i(s)$, $i = 1, \cdots r$, respectively.

Let us now consider a different choice of Λ and show that the simultaneous partial pole placement problem includes as a special case the classically well-known gain-margin problem.

Example 2.3: (The Gain Margin Problem): Let $g(s)$ be a single-input single-output plant of degree n. Let $\lambda g(s)$, $\lambda \in [a, b]$ where $a, b \in \mathfrak{R}$ be a family of plants. The problem is to design a compensator $k(s)$ such that the closed-loop system $\lambda g(s)[1 + \lambda k(s)g(s)]^{-1}$, $\lambda \in [a, b]$ have poles in $\mathfrak{S}_s$ for a given $\mathfrak{S}_s$. By choosing $\Lambda = [a, b]$, $g_\lambda(s) = \lambda g(s)$, $n_\lambda = n$, $d_\lambda = 0$, $\psi_\lambda = 1$, it follows that the above gain margin problem is a special case of the proposed problem.

Remark 2.4: By utilizing the flexibility in the choice of $\psi_\lambda(s)$, the designer on one extreme (i.e., by choosing $d_i = n_i + q$) can place the closed-loop poles of $g_\lambda(s)$, arbitrarily, and on the other extreme (i.e., by choosing $d_i = 0$) can restrict the closed-loop poles to lie in a suitably chosen open region $\mathfrak{S}_s$. However, in practice, one is frequently interested in placing a specified set of closed-loop poles while requiring that the unspecified poles are restricted to a given region $\mathfrak{S}_s$ of the complex plane. Such a requirement, we remark, may be met by selecting the integers d_i in the interval $[0, n_i + q]$, $i = 1, \cdots, r$. This remark serves to illustrate the design flexibility in the partial pole assignment problem.

III. Partial Pole Assignment of a Pair of Plants

Given a pair of distinct plants $g_1(s)$, $g_2(s)$ represented in their coprime factorization as $x_i(s)/y_i(s)$, $i = 1, 2$, respectively, where $x_i(s), y_i(s) \in H$. (Refer to [6], [8], and [10] for details on the coprime representation.) Given a pair of coprime biproper rational functions $\psi_1(s), \psi_2(s) \in H$, i.e., ψ_1, ψ_2 do not have a zero in common. Our first theorem concerns the simultaneous partial pole placement of two plants as follows.

Theorem 3.1: The pair of distinct plants $x_i(s)/y_i(s)$ is simultaneously partially pole assignable at the zeros in $\mathfrak{S}_u$ of $\psi_i(s)$, $i = 1, 2$, respectively, iff there exists $\Delta_1(s), \Delta_2(s) \in J$ such that the following holds.

1) $\Delta_1\psi_1 y_2 - \Delta_2\psi_2 y_1$ and $\Delta_2\psi_2 x_1 - \Delta_1\psi_1 x_2$ vanish at $s_1, s_2, \cdots, s_t \in \mathfrak{S}_u$ with multiplicities at least $m_1, m_2, \cdots, m_t$, respectively, where $s_1, \cdots, s_t$ are the zeros of $x_1y_2 - x_2y_1$ in $\mathfrak{S}_u$ with multiplicities $m_1, \cdots, m_t$, respectively.

2) If $x_1y_2 - x_2y_1$ vanishes at ∞ with multiplicity m_∞, then $\Delta_2\psi_2 x_1 - \Delta_1\psi_1 x_2$ vanishes at ∞ with multiplicity m_∞.

Proof: Let $x_c(s)/y_c(s)$ be the required compensator. A necessary and sufficient condition for solving the associated partial pole placement problem is the existence of $\Delta_1, \Delta_2 \in J$ such that

$$x_i(s)x_c(s) + y_i(s)y_c(s) = \Delta_i(s)\psi_i(s) \tag{3.1}$$

for $i = 1, 2$. Solving (3.1) for x_c and y_c, we have

$$x_c(s) = [\Delta_1\psi_1 y_2 - \Delta_2\psi_2 y_1]/[x_1y_2 - x_2y_1] \tag{3.2}$$

$$y_c(s) = [\Delta_2\psi_2 x_1 - \Delta_1\psi_1 x_2]/[x_1y_2 - x_2y_1]. \tag{3.3}$$

It follows that 1) is a necessary and sufficient condition for x_c, y_c to belong to H. Moreover, x_c and y_c do not have a common zero at s^* in $\mathfrak{S}_u$ for otherwise from (3.1) both ψ_1, ψ_2 vanish at s^* which contradicts the hypothesis that ψ_1, ψ_2 are coprime. Hence, x_c and y_c are coprime. Finally, 2) is a necessary and sufficient condition for $y_c(\infty) \neq 0$ and, hence, x_c/y_c is a proper rational function. Q.E.D.

The following corollary of Theorem 3.1 is immediate.

Corollary 3.2: Assume ψ_1, ψ_2 to be coprime and biproper, i.e., proper, but not strictly proper, and assume that the multiplicities of ∞ as a zero of x_1, x_2 and $x_1y_2 - x_2y_1$ are the same. The problem of partially pole assigning the pair of plants $x_i(s)/y_i(s)$ $i = 1, 2$ at the zeros in $\mathfrak{S}_u$ of $\psi_i(s)$, $i = 1, 2$, respectively, is equivalent to the problem of interpolating a self conjugate set of pairs of complex numbers (s_i, z_i), $i = 1, \cdots, \alpha$ by a rational function $\Delta_2(s)/\Delta_1(s)$ where $\Delta_2, \Delta_1 \in J$ and $\Delta_2/\Delta_1(\infty) \neq z_\infty$ for a given real z_∞.

Proof: Let s_i be a zero of $x_1y_2 - x_2y_1$ in $\mathfrak{S}_u$ of multiplicity 1. Then we have either of the following three possibilities.

1) s_i is a zero of $\psi_1 y_2$ and $\psi_2 y_1$. In this case the condition 1) of Theorem 3.1 is satisfied iff Δ_2/Δ_1 interpolates the pair of numbers

$$(s_i, \ \psi_1 x_2/\psi_2 x_1(s_i)).$$

2) s_i is a zero of $\psi_2 x_1$ and $\psi_1 x_2$. In this case, the condition 1) of Theorem 3.1 is satisfied iff Δ_2/Δ_1 interpolates the pair of numbers $(s_i, \psi_1 y_2/\psi_2 y_1(s_i))$.

3) s_i does not satisfy the condition 1) or 2). In this case the condition 1) of Theorem 3.1 is satisfied iff Δ_2/Δ_1 interpolates the pair of numbers $(s_i, \psi_1 y_2/\psi_2 y_1(s_i))$ which is the same as the pair of number $(s_i, \psi_1 x_2/\psi_2 x_1(s_i))$.

If s_i is a zero of $x_1y_2 - x_2y_1$ in $\mathfrak{S}_u$ of multiplicity > 1, the argument is similar and involves interpolation restrictions on Δ_2/Δ_1 with multiplicity. Thus, the condition 1) of Theorem 3.1 is equivalent to an interpolation condition on $\Delta_2/\Delta_1(s)$.

Finally, we need to show for the properness of the compensator that $y_c(\infty) \neq 0$. Equivalently, we need to show that the condition 2) of Theorem 3.1 is satisfied. Since ψ_1, ψ_2 do not vanish at ∞ and since the multiplicity of ∞ as a zero of x_1, x_2 and $x_1y_2 - x_2y_1$ are the same, the condition 2) of Theorem 3.1 is satisfied iff $\Delta_2/\Delta_1(\infty) \neq z_\infty = [(\psi_1 x_2)/(\psi_2 x_1)](\infty)$. Q.E.D.

Remark: The main contribution of this section is to show that the partial pole placement problem of a pair of plants may be posed as an interpolation condition on $\Delta_2(s)/\Delta_1(s)$. The restriction on the class of plant-pairs considered in Corollary 3.2 can in general be removed and one can use Theorem 3.1 analogously and describe the associated interpolation problem. It may be emphasized, however, that explicit construction of $\Delta_1(s)$ and $\Delta_2(s)$, satisfying the interpolation condition is necessary in order to synthesize the feedback compensator via (3.2), (3.3). This is now considered in the next section.

IV. The Interpolation Problem

Solution to the interpolation problem described in Corollary 3.2 under the special case $\mathfrak{S}_s = \mathfrak{S}^-$ has been obtained by Youla *et al.* [10]. In this section we state, without proof, a generalization of the interpolation lemma in [10] for an arbitrary $\mathfrak{S}_s$. The proof can be constructed, adapting techniques cited in [8, see ch. 2 and 3] and has been omitted.

Lemma 4.1 (Interpolation Lemma): Given a self-conjugate set of pairs of complex numbers (s_i, z_i), $i = 1, \cdots, t$. There exists $\Delta(s) \in J$ such that $\Delta(s_i) = z_i$ iff for every s_i, s_j which belong to the same connected component of $\mathfrak{S}_u \cap \mathfrak{R}$, the corresponding z_i, z_j have the same sign.

The main result of this section is to show that for a fixed t, even when the necessary and sufficient condition of Lemma 4.1 is satisfied there does not exist a $\Delta(s) \in J$ of *a priori* bounded degree which satisfies the interpolation constraint $\Delta(s_i) = z_i$.

Theorem 4.2: Given that $\mathcal{C}_s = \mathcal{C}^-$ and $t = 2$. Consider the pair of points $(1, 1)$, $(2, \lambda)$, $\lambda \in \mathfrak{R}$. A necessary condition that there exists a $\Delta(s) \in J$ of degress $\leq q$ such that $\Delta(1) = 1$, $\Delta(2) = \lambda$ is that

$$2^{-q} < \lambda < 2^q. \tag{4.1}$$

Proof: Let us write $\Delta(s) = \Sigma_{i=0}^q a_i s^i / \Sigma_{i=0}^q b_i s^i$. In order that $\Delta(s) \epsilon J$, one obtains the restriction $a_i, b_i > 0$ or $a_i, b_i < 0$ for $i = 0, \cdots, q$. A necessary and sufficient condition that $\Delta(s)$ satisfies the interpolation restriction is given as follows:

$$\text{positive row span of } \begin{bmatrix} 1 & 1 \\ 1 & 2^q \end{bmatrix} \tag{4.2}$$

$$\text{positive row span of } \begin{bmatrix} 1 & \lambda \\ 1 & 2^q \cdot \lambda \end{bmatrix}$$

is nonempty. (A positive row span of a matrix M is defined to be the set of all vectors νM where ν has all positive real entries.) The condition (4.2) is satisfied iff (4.1) is satisfied. The above claim follows trivially by visualizing the positive row spans as a convex set in $\mathfrak{R}^2$ and checking for conditions as to when the two convex sets in (4.2) intersect. Hence, the proof. Q.E.D.

As a consequence of Theorem 4.2 we have the following example.

Example 4.3: Assume $\mathcal{C}_s = \mathcal{C}^-$. Consider the one-parameter family of pairs of plants parameterized by $\lambda \in \mathfrak{R}$ given by

$$\frac{(\lambda+2)s+1}{[(-2\lambda-9)s+8](s+1)}, \quad \frac{\lambda s+3}{[(-2\lambda-5)s+4](s+1)}. \tag{4.3}$$

From the computation in Section III it follows that a necessary condition for the pair of plants (4.3) to be simultaneously stabilizable is the existence of $\Delta(s) \in J$ such that $\Delta(1) = 1$, $\Delta(2) = (2\lambda + 3)/(2\lambda + 5)$. Thus, from Theorem 4.2 if $(2\lambda + 3)/(2\lambda + 5) \leq 0$, the pair of plants is simultaneously unstabilizable. However, even when $(2\lambda + 3)/(2\lambda + 5) > 0$, as λ approaches -2.5 from the left, or -1.5 from the right, the minimum degree of the simultaneously stabilizing compensator increases unboundedly.

Remark 4.4: As this note was under review, an observation similar to the Example 4.3 has been reported in [7] using a different technique.

V. The Main Result

In this section, we consider a family F of single input single output proper plants of degree n given by

$$F = \{x_\lambda(s)/y_\lambda(s) | \lambda \in \Lambda,\ x_\lambda,\ y_\lambda \in H;\ x_\lambda,\ y_\lambda \text{ are coprime; } \deg x_\lambda/y_\lambda = n \text{ for all } \lambda\}. \tag{5.1}$$

Assume that F contains at least two distinct plants. Define $\eta_{ij}(s) = x_i y_j(s) - x_j y_i(s)$, $i, j \in \Lambda$. We now state the following main theorem.

Theorem 5.1: Let $x_1(s)/y_1(s)$, $x_2(s)/y_2(s)$ be two distinct plants in F. Let $\psi_\lambda(s)$, $\lambda \in \Lambda$ be a family of proper, but not strictly proper, rational functions with zeros only in $\mathcal{C}_u$ where $\psi_1(s)$, $\psi_2(s)$ are coprime. There exists a proper compensator which simultaneously partially pole assigns the plant $x_\lambda(s)/y_\lambda(s)$ at the zeros of $\psi_\lambda(s)$ $\lambda \in \Lambda$, respectively, iff there exists $\Delta_1(s)$, $\Delta_2(s) \in J$ such that the conditions 1) and 2) of Theorem 3.1 are satisfied together with the following additional condition.

For every $\lambda \epsilon \Lambda - \{1, 2\}$, the graph of $\Delta_1/\Delta_2(s)$ intersects the graph of $\psi_2\eta_{\lambda 1}/\psi_1\eta_{\lambda 2}$ at those points and only those points $s^* \in \mathcal{C}_u$ with multiplicity $m_0 - m_1$ where m_0 is the multiplicity of s^* as a zero of $\psi_\lambda \eta_{12}$ and m_1 is the multiplicity of s^* as a common zero of $\psi_1\eta_{\lambda 2}$ and $\psi_2\eta_{\lambda 1}$.

Proof: Let $x_c/y_c(s)$ be the required compensator. A necessary and sufficient condition that the compensator x_c/y_c simultaneously partially pole assigns the plants x_1/y_1 and x_2/y_2 at the zeros of ψ_1 and ψ_2, respectively, is given by the conditions 1) and 2) of Theorem 3.1. Hence, x_c and y_c can be solved using (3.2) and (3.3). Additionally, x_c/y_c simultaneously partially pole assigns every other plant x_λ/y_λ in F at ψ_λ, $\lambda \in \Lambda - \{1, 2\}$, respectively, iff

$$x_c x_\lambda + y_c y_\lambda = \Delta_\lambda \psi_\lambda. \tag{5.2}$$

From (3.2), (3.3), and (5.2), we obtain by eliminating x_c, y_c the following

$$\Delta_1 \psi_1 n_{\lambda 2} - \Delta_2 \psi_2 n_{\lambda 1} = \Delta_\lambda \psi_\lambda n_{12}. \tag{5.3}$$

It follows that a necessary and sufficient condition for the existence of $\Delta_\lambda \in J$, $\lambda \in \Lambda - \{1, 2\}$ to satisfy (5.3) is given by the condition 3) described above. Q.E.D.

Two easy corollaries of Theorem 5.1 are now stated. Let x_1/y_1, x_2/y_2 be a pair of distinct plants of degree n. Consider the family F_1 of plants

$$F_1 = \{g_\lambda(s) : g_\lambda = [\lambda x_1 + (1-\lambda)x_2]/[\lambda y_1 + (1-\lambda)y_2],\ \lambda \in [0, 1],\ \deg g_\lambda = n, \text{ for all } \lambda\}. \tag{6.4}$$

We have the following.

Corollary 5.2: Let $\mathcal{C}_s = \mathcal{C}^-$. The family of plants in F_1 is simultaneously stabilizable by a proper compensator iff there exists Δ_1, Δ_2, $\in J$ such that:

a) for every s^*, which is a zero of $x_1 y_2 - x_2 y_1$ in $\mathcal{C}^+$ with multiplicity m^*, $\Delta_1 x_2 - \Delta_2 x_1$, $\Delta_1 y_2 - \Delta_2 y_1$ vanish at s^* with multiplicity at least m^* and the multiplicity of ∞ as a zero of $\Delta_1 x_2 - \Delta_2 x_1$ is equal to the multiplicity of ∞ as a zero of $x_1 y_2 - x_2 y_1$.

b) Δ_1/Δ_2 does not intersect $\mathfrak{R}^- \triangleq \mathcal{C}^- \cap \mathfrak{R}$ at any point in $\mathcal{C}^+$.

Proof: The condition a) follows precisely from the conditions 1) and 2) of Theorem 3.1 assuming $\psi_1(s) \equiv \psi_2(s) \equiv 1$ and the pair of distinct plants being $g_1(s)$ and $g_2(s)$. The condition b) follows from the condition 3) of Theorem 5.1. The details are omitted. Q.E.D.

Corollary 5.3 (Gain Margin Problem): Let $\mathcal{C}_s = \mathcal{C}^-$. Assume $g(s) = x(s)/y(s)$ and $\Lambda = [a, b]$, $a < b$. The family of proper plants $\lambda g(s)$, $\lambda \in [a, b]$ is simultaneously stabilizable by a proper compensator iff there exists Δ_1, $\Delta_2 \in J$ such that the following holds.

1) If s^* is a zero of $g(s)$ in $\mathcal{C}^+$ with multiplicity m^*, then $\Delta_1/\Delta_2(s^*)$ equals 1 with multiplicity $\geq m^*$.

2) If s^* is a pole of $g(s)$ in $\mathcal{C}^-$ with multiplicity m^*, then $\Delta_1/\Delta_2(s^*)$ equals a/b with multiplicity $\geq m^*$.

3) If ∞ is a zero of $y(s)$ of multiplicity m_∞, then $\Delta_1/\Delta_2(\infty)$ equals a/b with multiplicity m_∞.

4) $\Delta_1/\Delta_2(s)$ does not intersect $\mathfrak{R}^-$ at any point in $\mathcal{C}^-$.

Proof: Follows trivially from Corollary 5.2 assuming $x_1(s) = ax(s)$, $x_2(s) = bx(s)$, and $y_1(s) = y_2(s) = y(s)$. Q.E.D.

Remark 5.4: Visualizing $\Delta_1/\Delta_2(s)$ as an analytic map from $\mathcal{C}^+$ to $\bar{\mathcal{C}} - (-\infty, 0]$, where $\bar{\mathcal{C}}$ is the complex plane together with the point at infinity; and noting that $\bar{\mathcal{C}} - (-\infty, 0]$ is conformally equivalent to the open unit disk, it follows that the interpolation problems in Corollaries 5.2 and 5.3 above can be tackled via classically well-known Nevanlinna–Pick interpolation methods. In particular, $\Delta_1/\Delta_2(s)$ can be constructed explicitly, if it exists, under the special cases mentioned in the corollaries. These have been detailed in [5] for the gain margin problem by Khargonekar and Tannenbaum and it is surprising that their methods actually extend to the simultaneous stabilization of a linear one-parameter family of plants F_1.

VI. Conclusion

To conclude, in this paper we have motivated the use of interpolation methods in the simultaneous partial pole assignment problems. Extension of these results to the multiinput multioutput systems is possible and partial results have been reported in [3].

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On the Sufficient Conditions for the Equality of the Unassignable Polynomial and Davison's Fixed Polynomial of Strongly Connected Systems

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Abstract—**In this technical note, it is established that the unassignable polynomial defined for a not strongly connected decentralized control system is not equal to Davison's fixed polynomial. This leads to a "sufficient condition" for the equality of the unassignable polynomial and Davison's fixed polynomial for strongly connected systems.**

I. Introduction

Corfmat and Morse in their paper proved that for a strongly connected decentralized control system, the unassignable polynomial $\mu_F^j(\lambda)$ for the triple (C_j, A_F, B_j), is equal to Davison's fixed polynomial (cf. [1, vide iii) Theorem 4, p. 490]). In the succeeding paragraph of the above theorem, they say, in particular, for a not strongly connected system, the above theorem need not necessarily hold. We feel that because of the vagueness regarding the "sufficient condition" of the above theorem, the results presented by Corfmat and Morse in their paper for a not strongly connected system, are not appealing as their procedure is computationally expensive.

This motivated the authors to prove the sufficiency part of the above theorem and use this result to obtain a computationally efficient procedure for splitting not strongly connected systems [3]. This is possible because the procedure discussed in [3] splits the not strongly connected system into s subsystems such that each subsystem is *complete.* As a consequence of this, the identification of fixed modes for a not strongly connected system, is made simpler.

In view of the above, in this correspondence, it is proved that the unassignable polynomial defined for a not strongly connected decentralized control system, is not equal to Davison's fixed polynomial. And a numerical example is included to illustrate the theory.

II. Statement of the Problem

Let $\Sigma \equiv \{C_i, A, B_i; K\}$ be a jointly controllable, jointly observable not strongly connected, and K channel linear system. Let $\mu_F^{j_i}(\lambda)$ denote the unassignable polynomial (computed from jth channel of ith strongly connected subsystem) of $\{C_j, A_F, B_j; K; i\}$ and $\alpha_\Sigma(\lambda)$ denote Davison's fixed polynomial of Σ defined by

$$\alpha_\Sigma(\lambda) = \Lambda \det(\lambda I - A_F) \tag{1}$$

and

$$A_F = (A + \sum_{i=1}^{K} B_i F_i C_i).$$

Then it is to be proved that

$$\mu_F^{j_i}(\lambda) \neq \alpha_\Sigma(\lambda) \tag{2}$$

where there are p number of strongly connected subsystems ($i = 1, 2, \cdots, p$) and ith strongly connected subsystem has q_i channels and $j \in q_i \subset K$.

III. Definitions

In this section, we present some definitions which we use in our proof in addition to the definitions borrowed from Corfmat and Morse [1].

Let $\alpha_\Sigma^i(\lambda)$ denote the fixed polynomial of ith strongly connected subsystem, so the fixed polynomial of the system is given by

$$\alpha_\Sigma(\lambda) = \prod_{i=1}^{p} \alpha_\Sigma^i(\lambda). \tag{3}$$

This is true since the closed-loop matrix for not strongly connected system is in upper block triangular structure (cf. [1, p. 493]).

Hence, we can obtain

$$Q^r(\lambda) = \det(\lambda I - \hat{A}_r^*) \tag{4}$$

and

$$\hat{Q}_r(\lambda) = \frac{Q^r(\lambda)}{\alpha_\Sigma^r(\lambda)}. \tag{5}$$

IV. Proof

Here we prove the relation given by the equation (2). It is known that for a not strongly connected system, the closed-loop matrix is in upper block triangular form and, therefore, we can write the following identity:

$$\mu_F^{j_i}(\lambda) = \alpha_\Sigma^i(\lambda) \prod_{\substack{r=1 \\ r \neq i}}^{p} Q^r(\lambda). \tag{6}$$

This is true because the unassignable polynomial computed from the jth channel of ith strongly connected subsystem consists of two components, viz.,

i) its own fixed polynomial $\alpha_\Sigma^i(\lambda)$,

and

ii) the characteristic polynomials of all other strongly connected subsystems

$$\left(\prod_{\substack{r=1 \\ r \neq i}}^{r=p} \det(\lambda I - \hat{A}_r^*)\right).$$

Equation (6) can be written as

$$\mu_F^{j_i}(\lambda) = \alpha_\Sigma^i(\lambda) \prod_{\substack{r=1 \\ r \neq i}}^{p} \hat{Q}_r(\lambda)\alpha_\Sigma^r(\lambda) \tag{7}$$

$$= \alpha_\Sigma(\lambda) \prod_{r=1}^{p} \hat{Q}_r(\lambda). \tag{8}$$

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