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TRANSCENDENTAL AND INTERPOLATION METHODS IN SIMULTANEOUS STABILIZATION AND SIMULTANEOUS PARTIAL POLE PLACEMENT PROBLEMS*

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Abstract. In this paper, we investigate the existence of a compensator which simultaneously renders a given r-tuple of multiinput multioutput $p \times m$ linear dynamical systems internally stable. In particular we parametrize a set of simultaneously stabilizable r-tuples of plants and show that provided $r \leqq \max(m, p)$, the above set is semialgebraic and dense in the space Σ of r-tuples of plants. Furthermore, we also consider an extension of the classical pole placement and stabilization problems and investigate the simultaneous partial pole placement problem. Consequently, we consider a suitable topology in Σ and obtain a necessary condition and a sufficient condition for the generic partial pole placement problem. Surprisingly enough, the problem of simultaneously stabilizing a triplet of $m \times m$ plants, chosen generically, is shown equivalent to the problem of partially pole placing a single $m \times m$ plant by a stable minimum phase compensator. On the other hand the problem of simultaneously stabilizing $m+2$ tuple of $1 \times m$ plants, chosen generically, is shown equivalent to the problem of partially pole placing a $1 \times m$ plant by a stable minimum phase compensator. Investigating the $1 \times m$ plants in details, we parametrize the set of all compensators simultaneously stabilizing a m tuple of $1 \times m$ plants chosen generically. Consequently, we obtain a necessary and sufficient condition for simultaneously stabilizing a $m+1$ tuple of $1 \times m$ plants chosen generically. Lastly we construct two numerical examples. The first example is a triplet of simultaneously unstabilizable single input single output plants, each of McMillan degree 1 that are simultaneously stabilizable in pairs. The second example is a triplet of 1×3 plants that are not simultaneously stabilizable.

Key words. partial pole placement, simultaneous stabilization, generic, semialgebraic

AMS(MOS) subject classifications. 14, 30, 93

1. Introduction. In order to introduce and analyze the problems considered in this paper we need the following notation.

$\mathbb{C}$:	the complex plane
$\mathbb{R}$:	the real line
$\mathbb{C}_s$:	a self-conjugate subset of $\mathbb{C}$ which intersects $\mathbf{R}$
$\mathbb{C}_u$:	$[\mathbb{C}-\mathbb{C}_s] \cup \{\infty\}$
$\mathbf{R}_u$:	$\mathbf{R} \cap \mathbb{C}_u$
$\mathbb{C}^-$:	open left half of the complex plane
H:	ring of proper rational functions with real co-efficients with poles in $\mathbb{C}_s$
$H^{p \times m}$:	set of $p \times m$ matrices whose elements belong to H
J:	set of multiplicative units in H
F:	quotient field of H [21, pp. 88-90].

Let us consider a set $G_1(s), \cdots, G_r(s)$ of r real, linear time invariant, proper $p \times m$ dynamical systems, each of a given fixed McMillan degree n_i, $i = 1, \cdots, r$ with m inputs and p outputs and ask the following problem.

Problem 1.1 (simultaneous stabilization). "When does there exist a nonswitching p input m output real, linear, time-invariant, proper compensator of arbitrary large but finite McMillan degree q, which stabilizes each of the r given plants either in discrete time or in continuous time?"

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The simultaneous stabilization problem 1.1 has been originally addressed by Birdwell, Castanon and Athans [2] for the case $m = p = 1$ and is followed up subsequently by Saeks and Murray [26] as a problem in reliability and in the design of a multi-mode control system. Subsequently, Vidyasagar and Viswanadham [34] addressed the above problem for the multiinput multioutput case and showed that if $\min(m, p) > 1$, a generic pair of $p \times m$ plants is simultaneously stabilizable in a suitably chosen topology. The case $m = p = 1$ is more difficult and it is shown [27], [34] that the problem of simultaneously stabilizing a pair of single input single output plants is equivalent to the well-known problem considered by Youla, Bongiorno and Lu [35]: When can a single plant be stabilized by a stable compensator? Moreover the problem of stabilizing a triplet of single input single output plant chosen generically has been shown by Ghosh [14] to be equivalent to the problem of partially pole placing a single input single output plant by a stable minimum phase compensator. The problem of simultaneous partial pole placement, introduced and analyzed for the single input single output plants in [14], consists in answering the following problem:

Problem 1.2 (simultaneous partial pole placement). "Given a r tuple $G_1(s), \cdots, G_r(s)$ of $p \times m$ proper transfer functions of degree n_i, $i = 1, \cdots, r$, respectively, together with a r tuple of proper rational functions $\psi_i(s)$ of degree d_i, $i = 1, \cdots, r$ respectively from H. Does there exist a proper $m \times p$ compensator $K(s)$ of degree $q \geqq \max_i [d_i - n_i]$ such that the closed loop systems $G_1(s)[I + K(s)G_1(s)]^{-1}, \cdots, G_r(s)[I + K(s)G_r(s)]^{-1}$ have, respectively, d_i poles precisely where $\psi_i(s)$ vanishes and all but the above d_i poles are in $\mathbb{C}_s$?"

As explained in [14], the partial pole placement problem 1.2 is an extension of the pole placement and stabilization problems. In fact, if we assume $d_1 = d_2 = \cdots = d_r = 0$ and $\mathbb{C}_s = \mathbb{C}^-$ then the problem 1.2 reduces to the simultaneous stabilization problem 1.1. On the other hand, if we choose $d_i = n_i + q$, one obtains the simultaneous pole placement problem described in [12] and [18]. Frequently however, it is necessary to choose d_i in between 0 and $n_i + q$ for all $i = 1, \cdots, r$. Such a choice, we remark, satisfies the need to shape the response of the closed loop system to the extent that the designer can place an arbitrary number of self conjugate poles in the closed loop while restricting the remaining poles in the region $\mathbb{C}_s$. The above problem 1.2 also includes the simultaneous version of the well-known dominant pole placement problem and the gain margin problem. It also appears in the analysis of a class of robust stabilization problems described in [15], [16], the reliable stabilization problem described in [11] and many other design problems described in [17].

The new ingredient in this paper is the application of interpolation and transcendental methods in analyzing the simultaneous partial pole placement problem. This is now described as follows.

Interpolation problem 1.3. "Given a self conjugate set of pairs of complex numbers (s_i, z_i), $i = 1, \cdots, t$, does there exist a $\Delta(s) \in J$ such that $\Delta(s_i) = z_i$, for all $i = 1, \cdots, t$?"

The problem of interpolation by a rational function is known to be very important in network theory and electrical engineering. For accounts of this see Youla-Saito [36], Zeheb-Lempel [39], Helton [20] and many other references therein. In control theory, similar interpolation methods have been successfully applied by Tannenbaum [30], Zames and Francis [37], Kimura [22] based on classical work of Nevanlinna [24] and Pick [25]. More recently, Vidyasagar and Davidson [32] have used interpolation methods to characterize the set of stable stabilizing compensators for a single input single output system.

Assuming $\mathbb{C}_s$ to be the open left half of the complex plane, the interpolation problem 1.3 has been posed and solved by Youla, Bongiorno and Lu [35] in order to analyze the stabilizability problem of a single input single output system by a stable compensator, in the continuous time. Complete solution to the problem 1.3 when $\mathbb{C}_s$ is arbitrary is given in [14]. Moreover generalizing the results due to Saeks and Murray [26], the problem of simultaneously stabilizing (partially pole assigning) a pair of single input single output plants is completely solved in [14]. Surprisingly however, as shown in [12], [13] and [14], in order to analyze the simultaneous stabilizability problem 1.1 for a triplet of single input single output systems, it is not enough to analyze the interpolation problem 1.3 and one asks the following transcendental problem.

Transcendental problem 1.4. "Given a t-tuple of rational functions $\delta_1(s), \cdots, \delta_t(s)$ from H, when does there exist a t tuple of rational functions $\Delta_1(s), \cdots, \Delta_t(s)$ in J such that

$$\Delta_1(s)\delta_1(s)+\cdots+\Delta_t(s)\delta_t(s)\equiv 0.\text{"} \tag{1.1}$$

As shown in [11], a triplet of single input single output plants $x_1(s)/y_1(s)$, $x_2(s)/y_2(s)$, $x_3(s)/y_3(s)$ chosen generically in the topology described in [3] is simultaneously stabilizable iff there exists a triplet of stable, minimum phase rational functions $\Delta_1(s)$, $\Delta_2(s)$, $\Delta_3(s)$ such that

$$\begin{aligned}\Delta_1(s)(x_1(s)y_3(s)-x_3(s)y_1(s))+\Delta_2(s)(x_2(s)y_3(s)-x_3(s)y_2(s))&\\ +\Delta_3(s)(x_2(s)y_1(s)-x_1(s)y_2(s))&\equiv 0.\end{aligned} \tag{1.2}$$

It may be remarked that (1.2) is satisfied just in case the plant $[x_1(s)y_3(s)-x_3(s)y_1(s)]/[x_2(s)y_3(s)-x_3(s)y_2(s)]$ is partially pole assignable at the zeros of $[x_1(s)y_2(s)-x_2(s)y_1(s)]$ in $\mathbb{C}_u$ by a stable, minimum phase compensator, Δ_1/Δ_2. Thus we remark that the partial pole assignability problem by a stable minimum phase compensator is an important problem in system theory which is addressed, among others, by the transcendental problem 1.4.

This paper is organized as follows. In § 2 we parametrize a set of simultaneously stabilizable r-tuples of plants and show that provided $r \leqq \max(m, p)$, the above set is semialgebraic and dense in the set of r-tuples of plants. In § 3 we analyze square systems, i.e. when $m=p$ and show that the problem of simultaneously stabilizing r tuples ($r \geqq 3$) of $m \times m$ systems chosen generically is equivalent to the problem of partially pole assigning a $r-2$ tuple of $m \times m$ systems by a stable, minimum phase compensator. Consequently, we obtain a necessary condition for the simultaneous partial pole placement of three or more square systems chosen generically. Finally in § 4 we discuss in considerable details the case when $\min(m, p)=1$, i.e. when the number of inputs or the number of outputs is one. In § 5 we construct two folklore examples which do not pre-exist in the literature. Section 6 concludes this paper with a discussion on the prospects of this rapidly growing simultaneous system design methodology.

2. A generic and/or semialgebraic parametrization of the partially pole assignable r-tuples of $p \times m$ systems.

2.1. We would first like to consider the basic mathematical setup, for the details of which we refer to Saeks et al. [26], [27], Vidyasagar et al. [33], [34] and Desoer et al. [9]. Every single input single output system, viewed as an element of F, can be written as $x(s)/y(s)$, where $x(s)$, $y(s) \in H$. Similarly, a $p \times m$ multiinput multioutput plant $G(s)$ has the left coprime representation $D(s)^{-1}N(s)$, where $N(s) \in H^{p \times m}$, $D(s) \in H^{p \times p}$. Moreover a r-tuple of m input p output plants can be written as

$$G_i(s)=D_i(s)^{-1}N_i(s), \qquad i=1, \cdots, r \tag{2.1}$$

where $N_i(s) \in H^{p\times m}$, $D_i(s) \in H^{p\times p}$, for all i. Of course if n_i is the McMillan degree of $G_i(s)$, $i=1,\cdots,r$ respectively, we consider the space Σ of r tuples of plants given by

$$\Sigma = \Sigma^{n_1}_{m,p} \times \cdots \times \Sigma^{n_r}_{m,p} \tag{2.2}$$

where

$$\Sigma^{n_i}_{m,p} = \{p\times m \quad G_i(s); \text{ degree } G_i(s) = n_i\}. \tag{2.3}$$

As has been described in [12], Σ is a quasiaffine algebraic variety in the affine space $\mathbb{R}^{(2n_1+1)mp} \times \cdots \times \mathbb{R}^{(2n_r+1)mp}$ and inherits the subspace topology.

Our main results of this paper concern the following pair of questions.

Question 2.1 (generic partial pole placement problem). Fix m, p, r and n_i. Let $\psi_1(s),\cdots,\psi_r(s) \in H$ be given. Is the set Σ_p of r-tuples $G_1(s),\cdots,G_r(s)$ which can be simultaneously partially pole assigned at the zeros of $\psi_1(s),\cdots,\psi_r(s)$ respectively by some compensator, open and dense in Σ?

Question 2.2 (semialgebraic parametrization). Does there exist a semialgebraic subset $\mathcal{S}$ in Σ which is open and dense in Σ_p?

A semialgebraic (see [4] for details) set is a finite union and intersection of sets defined by algebraic equations and inequations. It is a classical result by Tarski [31] and Seidenberg [28], that the property of being semialgebraic is preserved by a rational map. Anderson, Bose and Jury [1] have used this fact to show that the set of plants, of a given McMillan degree, which can be stabilized by some feedback gain is an open, semialgebraic subset in the space of all plants.

Before we state and prove the main results of this section, we refer to the following two well-known results, which also illustrate the significance of semialgebraicity. Routh and Hurwitz parametrized the set of monic polynomials of degree n in one variable, with real co-efficients that have roots in the open left half of the complex plane. They showed that the above set of polynomials is semialgebraic in $\mathbb{R}^n$. This result is surprising because their proof involves complex analytic methods. As a second application of semialgebraicity we note the following. Youla, Bongiorno and Lu [35] considered the problem of parametrizing the set of plants in $\Sigma^n_{m,p}$ which can be stabilized by a stable compensator. Their result shows that the above set may be described by an interlacing condition involving the poles and the blocking zeros of the plant on the nonnegative real axis. It is therefore a semialgebraic subset of $\Sigma^n_{m,p}$. This result is again surprising because the compensators under consideration is of a priori unbounded McMillan degree and therefore "decision-algebra" methods [1] are not applicable. In fact the parametrization problem involves solving a matrix analogue of the interpolation problem 1.3 which is once again a problem in "complex analysis."

At present a semialgebraic parametrization, of the space $\pi^t_{i=1}[\Sigma^{n_i}_{1,1} \cap H]$ of t tuples of rational functions $\delta_1(s),\cdots,\delta_t(s)$ which satisfy (1.1) for some $\Delta_1,\cdots,\Delta_t$ in J, does not exist. Thus it is unknown if Σ_p is semialgebraic. It is therefore of interest to answer Question 2.2. To clarify further, it is not enough to know that there exists a set of r-tuples of plants $\mathcal{S}$ that can be partially pole assignable by some compensator, where $\mathcal{S}$ is open and dense (generic) in Σ. It is also of interest to know if there exists a means to describe $\mathcal{S}$ as well—in particular, is $\mathcal{S}$ semialgebraic in Σ?

If $r=1$, every plant is pole assignable and is therefore partially pole assignable by some compensator. In this section we assume $r>1$ and prove the following:

THEOREM 2.4. *Assume $\psi_1(s),\cdots,\psi_r(s) \in H^{p\times p}$ such that* det $\psi_i(s)$, $i=1,\cdots,r$ *have zeros only in $\mathbb{C}_u$ and do not have a zero in common in $\mathbb{C}_u$. Assume*

$$1 < r \leqq \lfloor (m+p)/\min(m,p) \rfloor. \tag{2.4}$$

There exists an open, semialgebraic subset $\mathcal{S}$ *in* Σ *which is open and dense in* Σ_p, *the set of partially, pole assignable r-tuples of proper plants in* Σ. ($\lfloor n \rfloor$ *is defined to be the largest integer less than or equal to n.*)

Remark. In principle, one might obtain the semialgebraic set $\mathcal{S}$ using the elimination methods of Tarski [31] and Seidenberg [28]. Indeed, Byrnes and Anderson [6] have successfully used the concept of elimination, for the generic output feedback stabilization problem. For the purpose of computation, however, it is of interest to know explicitly the semialgebraic set $\mathcal{S}$, without going through "elimination" since it is known [10] to be computationally inefficient. Theorem 2.4 is a result on such an explicit parametrization as its proof would show.

It is of theoretical interest, however, to use "elimination methods" and prove the following:

THEOREM 2.5. *Assume* $r>1$, $\psi_i(s)$, $i=1,\cdots,r$ *belong to* $H^{p\times p}$ *and satisfy the condition mentioned in Theorem 2.4. A sufficient condition that there exists an open, dense, semialgebraic subset* $\mathcal{S}$ (*in* Σ) *of r-tuples of* $p\times m$ *proper plants* $G_1(s),\cdots,G_r(s)$ *of McMillan degrees* n_i, $i=1,\cdots,r$ *respectively which can be partially pole assigned at the zeros of* $\det\psi_i(s)$, $i=1,\cdots,r$ *respectively by an arbitrary large but finite proper compensator q is given by*

$$r\leqq \max(m,p). \tag{2.5}$$

Remark. Except for the case $\min(m,p)=1$ we note that the hypothesis (2.4) is stronger than (2.5).

It is interesting to note that by choosing $\det\psi_i(s)\equiv 1$, $i=1,\cdots,r$, we have the following corollary from Theorem 2.5.

COROLLARY 2.6 (Vidyasagar and Viswanadham [34]). *Assume* $\max(m,p)>1$. *A generic pair of* $p\times m$ *proper plants is simultaneously stabilizable.*

Note that the Theorems 2.4 and 2.5 describe only the sufficiency conditions. We now obtain the following necessary condition for the partial pole placement problem.

THEOREM 2.7. *Assuming* $r>1$, $\psi_i(s)$, $i=1,\cdots,r$ *as in Theorem 2.4. A necessary condition for generic simultaneous partial pole placement of a r-tuple of proper plants at the zeros of a generic r tuple of stable rational functions* $\det\psi_1(s),\cdots,\det\psi_r(s)$ *in* $\mathbb{C}_u$ *by some proper compensator of arbitrary large but finite McMillan degree q is given by*

$$q(m+p)+mp\geqq \sum_{i=1}^{r}\alpha_i \tag{2.6}$$

where α_i *is the number of zeros of* $\det\psi_i(s)$, $i=1,\cdots,r$ *respectively.*

Remark. The inequality (2.6) is not surprising and is precisely what one would guess from a "dimension count."

From Theorem 2.7 we now deduce the following.

COROLLARY 2.8. *Assume* $\alpha_i>q$, $r>1$, $\psi_i(s)$, $i=1,\cdots,r$ *as in Theorem 2.4. A necessary condition for generic simultaneous partial pole placement of a r-tuple of proper plants is given by*

$$r\leqq \max\left[mp/(\min_i \beta_i),\ m+p-1\right] \tag{2.7}$$

where $\alpha_i=q+\beta_i$, $i=1,\cdots,r$.

We remark that Corollary 2.8 includes the simultaneous pole placement problem as a special case when $\alpha_i=n_i+q$, $i=1,\cdots,r$. It is interesting to observe that (2.7) is independent of q and in particular we have the following.

COROLLARY 2.9 (Saeks and Murray [26], Vidyasagar and Viswanadham [34]). *Assume $m = p = 1$, $r = 2$. A generic pair of single input single output proper plants is not simultaneously pole assignable.*

Finally we note from Theorem 2.5 and Corollary 2.8 that the condition (2.5) is sharp in the following sense.

COROLLARY 2.10. *If* $\min(m, p) = 1$, $r > 1$ *and* $\alpha_i > q$ *and* $\psi_i(s)$, $i = 1, \cdots, r$ *as in Theorem* 2.4, *the inequality* (2.5) *is a necessary and sufficient condition for generic simultaneous partial pole placement of a r tuple of proper plants.*

To summarize the theorems and corollaries of this section, we make the following remark: If r is sufficiently small (given by (2.5)) a generic r tuple of multiinput and multioutput plants is simultaneously partially pole assignable. On the other hand under the assumptions of Corollary 2.8, if r is sufficiently large and fails to satisfy (2.7), a generic r tuple of multiinput multioutput plants is not simultaneously partially pole assignable. It may also be noted that for the generic simultaneous stabilization problem, one assumes $\alpha_i = 0$, $i = 1, \cdots, r$, so that Corollary 2.8 is inapplicable. In fact, no other necessary condition to the problem is known to exist. This constitutes an outstanding open problem in system theory.

We now sketch the proof of the theorems.

2.2. Proof of Theorem 2.4. First of all, we need to prove the following lemmas.

LEMMA 2.1. *Given a set of t vectors* $v_1, \cdots, v_t$ *in* $\mathbb{R}^{m+p}$ *with nonzero components. Then there exists another set of t vectors* $u_1, \cdots, u_t$ *such that* u_i *is orthogonal to* v_i, *for all* $i = 1, \cdots, t$ *and such that each component of* u_i *has the same sign for all* $i = 1, \cdots, t$ *respectively iff the set of vectors* $v_1, \cdots, v_t, -v_1, \cdots, -v_t$ *misses an orthant in* $\mathbb{R}^{m+p}$.

Proof. (If part). Let θ be the orthant which does not contain the vectors $v_1, \cdots, v_t$, $-v_1, \cdots, -v_t$. For a given vector v_i belonging to the orthant θ_1 it is well known that there exists a vector orthogonal to v_i in the orthant θ, since θ is different from θ_1, for all $i = 1, \cdots, t$. Thus in particular it is possible to choose $u_1, \cdots, u_t \in \theta$ such that $u_i \cdot v_i = 0$, $i = 1, \cdots, t$.

(Only if part). Assume that the set of vectors $v_1, \cdots v_t, -v_1, \cdots, -v_t$ does not miss any orthant in $\mathbb{R}^{m+p}$. It is clear that there does not exist any orthant θ in $\mathbb{R}^{m+p}$ such that $u_i \in \theta$ and $u_i \cdot v_i = 0$, for otherwise there would exist two nonzero vectors in the same orthant θ of $\mathbb{R}^{m+p}$ that are orthogonal to each other, which is absurd. Thus there does not exist a set of t vectors $u_1, \cdots, u_t$ with the above mentioned property. Q.E.D.

LEMMA 2.2. *Given a self conjugate set of tuples* (s_i, M_i), $i = 1, \cdots, t$ *where* $s_i \in \mathbb{C}$ *and* M_i *is a* $p \times p$ *complex matrix,* $i = 1, \cdots, t$ *respectively. Moreover for all* $i \in \{1, \cdots, t\}$ *for which* $s_i \in \mathbb{R}_u$, *assume that* $\det M_i > 0$. *There exists* $\Delta(s) \in H^{p\times p}$, $\det \Delta(s) \in J$ *such that*

$$\Delta(s_i) = M_i \tag{2.8}$$

for all $i = 1, \cdots, t$.

Remark. The main idea of Lemma 2.2 is originally due to Vidyasagar and Viswanadham [34] and the proof that we now sketch is an adaptation of their procedure.

Proof. Assume first of all that $\mathbb{C}_s$ contains the open left half of the complex plane. Let us define $N_p(s) \in H^{p\times p}$ such that

$$N_p(s_j) = 0, \qquad j = 1, \cdots, t \tag{2.9}$$

and that $N_c(s)$ has no other blocking zero in $\mathbb{C}_u$. A blocking zero is a point where the matrix function $N_p(s)$ vanishes as a matrix. Let $D_p(s)$ be such that

$$D_p(s_j) = M_j, \qquad j = 1, \cdots, t \tag{2.10}$$

and that $N_p(s)$, $D_p(s)$ are co-prime. It has been shown by Youla, Bongiorno and Lu [35] that the plant $N_p(s)D_p(s)^{-1}$ is stabilizable by a stable compensator, since at the zeros s_j, $j=1,\cdots,t$ of $N_p(s)$ in $\mathbb{C}_u$, $\det D_p(s_j)=\det M_j$, $j=1,\cdots,t$ have the same sign. Thus there exists $N_c(s)$, $D_c(s)$, $\Delta_1(s)\in H^{p\times p}$, $\det D_c(s)\in J$, $\det \Delta_1(s)\in J$ such that

$$N_c(s)N_p(s)+D_c(s)D_p(s)=\Delta_1(s). \tag{2.11}$$

Clearly we have

$$D_c^{-1}(s_j)\Delta_1(s_j)=D_p(s_j)=M_j, \qquad j=1,\cdots,t. \tag{2.12}$$

Defining $\Delta(s)=D_c^{-1}(s)\Delta_1(s)$, we have the required matrix function $\Delta(s)$.

In general if $\mathbb{C}_s$ does not contain the open left half of the complex plane, assume that there exists ball B_ε of radius ε with center at $c_1\in\mathbb{R}$ contained in $\mathbb{C}_s$. Note that since $\mathbb{C}_s$ intersects $\mathbb{R}$, such a ball would always exist. Let us now conformally transform B_ε onto the open left half of the complex plane by the map

$$\psi: z\mapsto\frac{s-c_1+\varepsilon}{s-c_1-\varepsilon}. \tag{2.13}$$

Define $z_j=(s_j-c_1+\varepsilon)/(s_j-c_1-\varepsilon)$, $j=1,\cdots,t$ and construct $\Delta(z)\in H^{p\times p}$, $\det\Delta(z)\in J$ such that $\Delta(z_j)=M_j$, $j=1,\cdots,t$, $\det\Delta(1)\neq 0$. It follows that $\Delta[(s-c_1+\varepsilon)/(s-c_1-\varepsilon)]$ is the required matrix function. Q.E.D.

LEMMA 2.3. *Given a pair of nonzero vectors* $(\alpha_1,\cdots,\alpha_p)$, $(\beta_1,\cdots,\beta_p)\in\mathbb{R}^p$, $p>1$ *there exists a* $m\times m$ *matrix* M *such that*

$$(\alpha_1,\cdots,\alpha_p)M=(\beta_1,\cdots,\beta_p) \tag{2.14}$$

and

$$\det M>0. \tag{2.15}$$

Proof. Let the ijth entry of M be given by a_{ij}. We may expand $\det M$ as

$$\det M=\sum_{i=1}^{p} a_{1i}\Delta_{1i} \tag{2.16}$$

where Δ_{1i} is the adjoint of the $1i$th entry in M. Assume without any loss of generality that $\alpha_1\neq 0$. One can rewrite (2.14) as follows:

$$a_{1i}=-\sum_{j=2}^{p}\frac{\alpha_j}{\alpha_1}a_{ji}+\frac{1}{\alpha_1}\beta_i. \tag{2.17}$$

Substituting a_{1i} in (2.16), we may check by straightforward computation that

$$\det M=\frac{1}{\alpha_1}\sum_{i=1}^{p}\beta_i\Delta_{1i}. \tag{2.18}$$

Clearly for a given α_1 and $\beta_1,\cdots,\beta_p$ we can write (2.18) as

$$\det M=a_{21}a'+a'' \tag{2.19}$$

where a', a'' are nonzero multinomial expressions in a_{ij}, $i=2,\cdots,p$; $j=1,\cdots,p$; independent of a_{21}. For any choice of $a_{22},\cdots,a_{2p},a_{31},\cdots,a_{3p},\cdots,a_{pp}$, such that $a'\neq 0$, it is possible to choose a_{21} such that $\det M>0$. The entries $a_{11},\cdots,a_{1p}$ are to be computed using (2.17) completing the proof. Q.E.D.

We would now proceed to prove Theorem 2.4. Assume $p \leqq m$ without any loss of generality. Let the r tuple of plants be given by (2.1). Let us represent the compensator as

$$N_c(s)D_c(s)^{-1} \tag{2.20}$$

where $N_c(s) \in H^{m \times p}$, $D_c(s) \in H^{p \times p}$, $N_c(s)$, $D_c(s)$ are coprime. It is well known that the compensator (2.20) partially pole assigns the r-tuple of plants (2.1) at the zeros of det $\psi_i(s)$, $i = 1, \cdots, r$ iff

$$N_i(s)N_c(s) + D_i(s)D_c(s) = \psi_i(s)\Delta_i(s) \tag{2.21}$$

for some $\Delta_i(s)$, $\psi_i(s) \in H^{p \times p}$, $\det \Delta_i(s) \in J$, $i = 1, \cdots, r$ respectively. We may write (2.21) in matrix form as

$$\begin{bmatrix} N_1(s) & D_1(s) \\ \vdots & \vdots \\ N_r(s) & D_r(s) \end{bmatrix} \begin{bmatrix} N_c \\ \\ D_c \end{bmatrix} = \begin{bmatrix} \psi_1(s)\Delta_1(s) \\ \vdots \\ \psi_r(s)\Delta_r(s) \end{bmatrix}. \tag{2.22}$$

Let us denote the $(rp) \times (m+p)$ matrix in the left-hand side of (2.22) by $M(s)$. Let us also define a set of r-tuples of plants $(G_1, \cdots, G_r)$ as follows.

$$\Omega = \{(G_1, \cdots, G_r) | M(s) \text{ in (2.22) has full rank for all points in } \mathbb{C}\}. \tag{2.23}$$

Note first of all that Ω can be defined as union and intersection of sets of the type

$$f_\alpha(g_1, \cdots, g_r) \neq 0 \tag{2.24}$$

where $f_\alpha(\cdot)$ are polynomials in the coefficients of $g_1(s), \cdots, g_r(s)$. Since $f_\alpha \not\equiv 0$, (2.24) describes a union and intersection of open and dense subsets of Σ. Thus Ω is open, semialgebraic and dense in Σ, the space of r-tuples of plants. We now show that when $r < \lfloor (m+p)/\min(m,p) \rfloor$, $\Omega \subset \Sigma_p$ so that it is enough to define the required set $\mathcal{S} = \Omega$. On the other hand we consider the case $\min(m,p) > 1$, $r \min(m,p) = m+p$ and the case $\min(m,p) = 1$, $r = m+p$ and show the existence of an open, semialgebraic and dense subset $\mathcal{S}$ of Σ_p.

Case 1 ($r < \lfloor (m+p)/\min(m,p) \rfloor$). Since H is a principal ideal domain [38], for every r tuple of plants in Ω, $M(s)$ has a right inverse $M'(s)$ in $H^{(m+p) \times rp}$ [33] provided $rp \leqq m+p$. Thus for a r-tuple of plants in Ω, (2.22) is indeed solvable for $N_c(s)$, $D_c(s)$. We need to check, however, that the solution would yield $N_c(s)$, $D_c(s)$ which are coprime and that the compensator $N_c(s)D_c(s)^{-1}$ is proper.

Assume that $N_c(s)$, $D_c(s)$ are not coprime. It follows that there exists $s^* \in \mathbb{C}_u$ such that rank $[N_c^T D_c^T]^T(s^*) < p$. It follows from (2.22) that $\det[\psi_i(s^*)\Delta_i(s^*)] = 0$ for $i = 1, \cdots, r$. Since $\det \Delta_i(s) \in J$, it follows that $\det[\psi_i(s^*)] = 0$ for $i = 1, \cdots, r$ which contradicts the assumption on $\psi_i(s)$, $i = 1, \cdots, r$. Hence $N_c(s)$, $D_c(s)$ are coprime.

To see that $N_c(s)D_c(s)^{-1}$ is proper, it is enough to show that $\det D_c(\infty) \neq 0$. We claim, first of all, that for every r-tuple of plants in Ω, the associated matrix $M(s)$ has a right inverse $M^1(s)$ with the property that if $[P_1(s) \cdots P_r(s)]$ is the last p row of $M^1(s)$, (where $P_i(s)$, $i = 1, \cdots, r$ are $p \times p$ matrices), $\det P_i(\infty) \neq 0$ for all $i = 1, \cdots, r$. First of all, let us assume the claim. Since $\det \psi_j(\infty) \neq 0$ for some $j = 1, \cdots, r$, it follows that $\psi_j(\infty)\Delta_j(\infty)$ is an arbitrary $p \times p$ matrix for some $j = 1, \cdots, r$ and for an arbitrary choice of $\Delta_j(s)$. Since $D_c(s) = \sum_{k=1}^r P_k(s)\psi_k(s)$, it follows from (2.22) that for a suitable choice of $\Delta_1(s), \cdots, \Delta_r(s)$, $\det D_c(\infty) \neq 0$.

Now we proceed to prove the claim. If $M^1(s)$ does not satisfy the above property, let P be a $(m+p) \times rp$ real matrix which satisfies the above property. Let us define $M^1_\varepsilon(s) = M(s) + \varepsilon P$ where ε is a nonzero real number. Clearly for all but finitely many values of ε, $M_\varepsilon^1(s)$ satisfies the above property. Moreover for small ε we have $\det[M(s)M_\varepsilon^1(s)] \in J$ since by construction $\det[M(s)M^1(s)] \in J$. Thus there exists $\varepsilon = \varepsilon^*$ such that $M_{\varepsilon^*}^1(s)$ is the required right inverse.

Case 2 ($\min(m, p) > 1$, $r \min(m, p) = m + p$). Assume without any loss of generality the case $p \leqq m$ and consider the worst case when $rp = m + p$. The case $rp < m + p$ is analogous to Case 1 above. Note that the matrix $M(s)$ in (2.22) is square and we can write

$$[N_c(s)^T D_c(s)^T]^T = M(s)^{-1}[\Delta_1^T \psi_1^T \cdots \Delta_r^T \psi_r^T]. \tag{2.25}$$

We now define the required subset $\mathscr{S}$ of Ω as follows:

$$\mathscr{S} \triangleq \{(G_1, \cdots, G_r) | \det M(s) \text{ has simple zeros at } s_1, \cdots, s_t \text{ in } \mathbb{C} \text{ and does not vanish at } \infty;\ \operatorname{Adj} M(s_i),\ i = 1, \cdots, t \text{ are of rank 1; if the rows of } \operatorname{Adj} M(s_i) \text{ are spanned by the nonzero vector } [\mathbf{c}_{i1}, \cdots, \mathbf{c}_{ir}] \text{ then } \mathbf{c}_{i,j}\psi_j(s_i) \neq \mathbf{0}, \quad i = 1, \cdots, t; j = 1, \cdots, r.\} \tag{2.26}$$

It is clear that $\mathscr{S}$ is open, dense and semialgebraic in Ω. Moreover, in order that $N_c(s) \in H^{m \times p}$, $D_c(s) \in H^{p \times p}$, it is necessary and sufficient that

$$\operatorname{Adj} M(s_i)[\Delta_1^T \psi_1^T(s_1), \cdots, \Delta_r^T \psi_r^T(s_i)]^T = 0 \tag{2.27}$$

for all s_i where $\det M(s_i) = 0$, $i = 1, \cdots, t$. It follows from the definition of $\mathscr{S}$ in (2.26) that a necessary and sufficient condition for the solvability of (2.27) is that

$$\sum_{j=1}^{r} \mathbf{c}_{ij}\psi_j(s_i)\Delta_j(s_i) = 0 \tag{2.28}$$

for $i = 1, \cdots, t$. That indeed (2.28) is satisfied by a set of $\Delta_j(s)$, $j = 1, \cdots, r$ follows from Lemma 2.2 and Lemma 2.3.

Case 3 ($\min(m, p) = 1$, $r = m + p$). The computation of Case 2 can be repeated so that we obtain an equation of the type (2.28). Since $\mathbf{c}_{ij}\psi_j(s_i) \neq 0$, it follows from Lemmas 2.1 and 2.2 that indeed there exist $\Delta_1, \cdots, \Delta_r(s)$ which satisfy equation (2.28), provided for $s_i \in \{s_1, \cdots, s_t\} \cap \mathbb{R}$, $i = 1, \cdots, t_1$ the following condition is satisfied: Let us define $\{s_1, \cdots, s_{t_1}\} \triangleq \{s_1, \cdots, s_t\} \cap \mathbb{R}_u$. The real vectors

$$v_i = [\mathbf{c}_{i,1}\psi_1(s_i), \cdots, \mathbf{c}_{i,r}\psi_r(s_i)] \tag{2.29}$$

for $i = 1, \cdots, t_1$ and $-v_i$, $i = 1, \cdots, t_1$ miss an orthant in $\mathbb{R}^{m+p}$. Since "missing an orthant" in $\mathbb{R}^{m+p}$ may be described by open semialgebraic conditions, it follows that the partially pole assignable r tuples of plants in $\mathscr{S}$ is open, semialgebraic. Q.E.D.

Remark. The distinction between the results in Case 2 and Case 3 of Theorem 2.4 is to be noted. In Case 2, we claim that every r tuple of plants in the semialgebraic set $\mathscr{S}$ is simultaneously partially pole assignable. In Case 3, we only claim that the set of simultaneously partially pole assignable plants in $\mathscr{S}$ is semialgebraic and is given precisely by the above condition of "missing an orthant."

2.3. Proof of Theorem 2.5. The case $\min(m, p) = 1$ follows from Case 1 of Theorem 2.4. The case $\min(m, p) > 1$ proceeds by a reduction to the case $\min(m, p) = 1$ by a procedure called "vectoring down" adopted from Stevens' thesis [29] and from Brasch-Pearson [5]. The following two lemmas are now stated without proof (see [12], [18] for a proof).

LEMMA 2.4. *Given a r-tuple of $p \times m$ plants $G_i(s)$ of degrees n_i, each with n_i simple poles. There is an open dense set of $1 \times p$ vectors $v \in \mathbb{R}^p$ such that $vG_i(s)$ has degree n_i, for all i.*

LEMMA 2.5. *Given a r-tuple of* $p \times m$ *plants* $G_i(s)$, $i = 1, \cdots, r$. *There exists a constant gain output feedback* K *such that the closed-loop systems* $G_i(s)[I + KG_i(s)]^{-1}$ *have distinct simple poles.*

Thus by choosing any $(v, K) \in \mathbb{R}^p \times \mathbb{R}^{mp}$, we have a mapping

$$\phi: \Sigma \times \mathbb{R}^p \times \mathbb{R}^{mp} \to \Sigma_1, \tag{2.30}$$

$$\phi(G_1(s), \cdots, G_r(s), v, k) = (vG_i(s)[I + KG_i(s)]^{-1})_{i=1}^r \tag{2.31}$$

where Σ is the space of r-tuples of $p \times m$ plants and Σ_1 is the space of r-tuples of $1 \times m$ plants.

Since ϕ is rational in the coefficients of $G_1, \cdots, G_r, v, K$; the inequalities (2.24) together with the mapping ϕ would define semialgebraic, open, dense subset Ω_1 of $\Sigma \times \mathbb{R}^p \times \mathbb{R}^{mp}$ given by

$$f_\alpha(vG_1[I + KG_1]^{-1}, \cdots, vG_r[I + KG_r]^{-1}) \neq 0. \tag{2.32}$$

By considering the projection

$$\text{Proj}: \Sigma_{m,p}^{n_1} \times \cdots \times \Sigma_{m,p}^{n_r} \times \mathbb{R}^p \times \mathbb{R}^{mp} \to \Sigma_{m,p}^{n_1} \times \cdots \times \Sigma_{m,p}^{n_r} \tag{2.33}$$

we have the set

$$\mathcal{S} \triangleq \text{Proj}\, \Omega_1. \tag{2.34}$$

By the Tarski [31]-Seidenberg [28] theory of elimination over $\mathbb{R}$, $\mathcal{S}^1$ is semialgebraic. Moreover $\mathcal{S}^1$ is dense, since Ω_1 is dense. Thus $\mathcal{S}^1$ is semialgebraic, dense and is defined by union and/or intersection of sets given by polynomial equations or inequality

$$g_\alpha > 0 \quad \text{and} \quad [g_{\beta_1} > 0 \text{ or } g_{\beta_2} = 0] \quad \text{and} \quad g_\gamma = 0. \tag{2.35}$$

Since $g_\gamma(\mathcal{S}^1) = 0 \Rightarrow g_\gamma \equiv 0$, hence $\mathcal{S}$ defined by union and/or intersection of sets of the type $[g_\alpha > 0$ and $g_{\beta_1} > 0]$ is semialgebraic, open and dense in Σ. Note that $\mathcal{S}$ is dense in $\mathcal{S}^1$ since $\mathcal{S}$ is obtained by removing the set $[g_{\beta_2} \neq 0]$ from $\mathcal{S}^1$.

Thus a max(m, p) tuple of plants in $\mathcal{S}$ can be partially pole assigned by the vectoring down technique. Q.E.D.

2.4. Proof of Theorem 2.7. Let $G_1(s), \cdots, G_r(s)$ be the given r tuple of proper plants in Σ. Let $K(s)$ be a compensator in $\Sigma_{p,m}^q$. The associated return difference equation, $\det[I + K(s)G_i(s)] = 0$ is given by

$$\Pi_i(s) \triangleq \sum_{j=0}^{n_i+q} c_{ij}s^j \quad \text{for all } i = 1, 2, \cdots, r. \tag{2.36}$$

A generic r-tuple of plants defines a smooth mapping χ, between the compensator parameters and the coefficient of the return difference polynomials given by

$$\chi: \Sigma_{p,m}^q \to \mathbb{RP}^{n_1+q+1} \times \cdots \times \mathbb{RP}^{n_r+q+1}, \tag{2.37}$$

$$\chi(K) = ([c_{1,0}, \cdots, c_{1,n_1+q}], \cdots, [c_{r,0}, \cdots, c_{r,n_r+q}]) \tag{2.38}$$

where the right-hand side of (2.38) has been defined in homogeneous co-ordinates. It is well known [7], [8], [19] that $\Sigma_{p,m}^q$ is a manifold of dimension $q(m+p) + mp$. To say that there exists a compensator which partially pole assigns a generic r set of α_i self conjugated poles is to say that image χ contains a $\sum_{i=1}^r \alpha_i$ dimensional submanifold of $\mathbb{RP}^{n_1+q+1} \times \cdots \times \mathbb{RP}^{n_r+q+1}$. By Sard's theorem [23] a necessary condition is given by (2.6) concluding the proof. Q.E.D.

2.5. Proof of the corollaries. The Corollary 2.6 is immediate from Theorem 2.5 assuming $\psi_i(s) \equiv 1, i = 1, \cdots, r$. We now prove the Corollary 2.8. Substituting $\alpha_i = q + \beta_i$ in (2.6), we have

$$q(m+p-r)+mp \geqq \sum_{i=1}^{r} \beta_i \geqq r \min_i \beta_i. \tag{2.39}$$

A necessary condition that there exists some $q \in \mathbb{N}$ satisfying (2.39) is given by

$$r \leqq m+p-1 \quad \text{or} \quad r \min_i \beta_i \leqq mp. \tag{2.40}$$

Thus (2.7) is indeed a necessary condition concluding the proof of Corollary 2.8. Corollary 2.9 is a consequence of Corollary 2.8. Finally, the Corollary 2.10 follows from Theorems 2.5, 2.7 and Corollary 2.8.

3. Simultaneous partial pole placement of square systems.

3.1. The purpose of this section is to analyze the partial pole placement of a square system and obtain various equivalent characterization of the problem. Results of this type has been originally obtained by Saeks and Murray in [26] where they have shown that the problem of simultaneously stabilizing a pair of single input single output plants is equivalent to the problem of stabilizing a plant by a stable compensator. Likewise, it has been shown by Vidyasagar and Viswanadham [34] that the problem of simultaneously stabilizing r multiinput multioutput plants is equivalent to the problem of stabilizing $r-1$ plants by a stable compensator.

In this section we make contact with the "matrix version" of the transcendental problem described in the introduction. Adapting the proof of the "strong stabilization problem" in [35], we obtain a necessary condition for the solvability of the transcendental problem. We remark, however, that a necessary and sufficient condition for the transcendental problem is unknown, although for the scalar case, a separate necessary condition and a sufficient condition has been reported in [14]. The necessary condition also appears in the proof of Theorem 2.4 (Case 3). An interesting zero interlacing property of the necessary condition is described in § 5.

Let us now prove the following:

THEOREM 3.1. *A necessary and sufficient condition for the simultaneous partial pole placement of a r tuple* $(r \geqq 3)$ *of* $m \times m$ *plants* $G_1(s), \cdots, G_r(s)$, *chosen generically, at the zeros of* $\det \psi_1(s), \cdots, \det \psi_r(s)$ *in* $\mathbb{C}_u$ *by a nonswitching* $m \times m$ *compensator is given by the existence of* $\Delta_1(s), \cdots, \Delta_r(s) \in H^{m\times m}$, $\det \Delta_i(s) \in J$, $i = 1, \cdots, r$ *satisfying*

$$[\Delta_1\psi_1(s) \quad \Delta_2\psi_2(s)]\begin{bmatrix} N_1(s) & N_2(s) \\ D_1(s) & D_2(s)\end{bmatrix}^{-1}\begin{bmatrix} N_i(s) \\ D_i(s)\end{bmatrix} = \Delta_i(s)\psi_i(s) \tag{3.1}$$

for $i = 3, \cdots, r$. *Here* $G_i(s) = N_i(s)D_i(s)^{-1}$ *is a coprime fraction representation of the ith plant and* $\psi_1(s), \cdots, \psi_r(s) \in H^{m\times m}$ *with the property that* $\det \psi_i(s)$, $i = 1, \cdots, r$ *do not have a common zero in* $\mathbb{C}_u$ *and have zeros only in* $\mathbb{C}_u$.

Remark. Let us define

$$[\mathcal{K}_{i1}^T(s) \quad \mathcal{K}_{i2}^T(s)]^T \triangleq \left\{\operatorname{Adj}\begin{bmatrix} N_1(s) & N_2(s) \\ D_1(s) & D_2(s)\end{bmatrix}\right\}\begin{bmatrix} N_i(s) \\ D_i(s)\end{bmatrix}, \tag{3.2}$$

$$\mathcal{K}_3(s) = \det\begin{bmatrix} N_1(s) & N_2(s) \\ D_1(s) & D_2(s)\end{bmatrix} \tag{3.3}$$

and rewrite (3.1) as

$$\Delta_1\psi_1\mathcal{K}_{i1} + \Delta_2\psi_2\mathcal{K}_{i2} = \Delta_i\psi_i\mathcal{K}_3, \tag{3.4}$$

$i=3,\cdots,r$. The matrix version of the transcendental Problem 1.4 is to solve (3.4) for a suitable $\Delta_i(s)$, where $\det \Delta_i(s) \in J$, $i=1,\cdots,r$. We now state the following necessary condition for the solvability of (3.4).

THEOREM 3.2. *Let $s_1,\cdots,s_t$ be a set of blocking zeros of $\psi_{i_0}(s)$, $i_0 \in \{3,\cdots,r\}$ in a connected component of $\mathbb{R}_u$. A necessary condition that the r-tuple of plants is simultaneously partially pole assignable at the zeros of* $\det \psi_i(s)$ *in $\mathbb{C}_u$, $i=1,\cdots,r$ respectively (where* $\det \psi_i(s)$, *$i=1,\cdots,r$ do not have a common zero in $\mathbb{C}_u$) is given by*

(3.5) "sign $[\det \psi_1 \mathcal{K}_{i_0 1}(s_j)] \times [\det \psi_2 \mathcal{K}_{i_0 2}(s_j)$ *is the same for* $j=1,\cdots,t$."

3.2. Proof of Theorem 3.1. Let $K(s)=D_c(s)^{-1}N_c(s)$ be the coprime representation of the required compensator. (only if): Assume simultaneous partial pole assignability of the r-tuple of plants $G_1,\cdots,G_r$ by the compensator $K(s)$. Clearly there exists $\Delta_1(s),\cdots,\Delta_r(s) \in H^{m\times m}$, $\det \Delta_i(s) \in J$, $i=1,\cdots,r$ respectively such that

$$N_c(s)N_i(s)+D_c(s)D_i(s)=\Delta_i(s)\psi_i(s) \tag{3.6}$$

for all $i=1,\cdots,r$. Eliminating $D_c(s)$ and $N_c(s)$ from (3.6), we obtain (3.1).

(if): Assume that there exists $\Delta_1(s),\cdots,\Delta_r(s) \in H^{m\times m}$, $\det \Delta_i(s) \in J$ satisfying (3.1). Let us denote

$$P(s)=\begin{bmatrix} N_1(s) & N_2(s) \\ D_1(s) & D_2(s)\end{bmatrix}, \tag{3.7}$$

$$[N_c(s) \quad D_c(s)] \triangleq [\Delta_1\psi_1(s) \quad \Delta_2\psi_2(s)]P(s)^{-1}. \tag{3.8}$$

We need to show that $N_c(s), D_c(s) \in H^{m\times m}$; $N_c(s)$, $D_c(s)$ are coprime, $N_c(s)$, $D_c(s)$ satisfy (3.6) and finally $D_c^{-1}N_c(s)$ is proper. Assume generically that

(3.9) a. $P(s)$ has simple zeros in $\mathbb{C}_u$ at $s_1,\cdots,s_t$,

(3.10) b. Adj $P(s_i)$, $i=1,\cdots,t$ are of rank 1,

(3.11) c. If the rows of Adj $P(s_i)$ are spanned by $[r_{i,1}\cdots r_{i,2m}]$, then

$$[r_{i1} \quad r_{i,2m}][N_j^T \quad D_j^T]^T(s_i)\neq 0 \quad \text{for } j=3,\cdots,r,$$

(3.12) d. $\det P(\infty)\neq 0$.

To see that $N_c(s), D_c(s) \in H^{m\times m}$, we consider the following. From (3.1), (3.9) it follows that

$$[\Delta_1\psi_1(s_1) \quad \Delta_2\psi_2(s_i)]\,\text{Adj}\,P(s_i)[N_j(s_i)^T \quad D_j(s_i)^T]^T=0 \tag{3.13}$$

for $i=1,\cdots,t$; $j=3,\cdots,r$. From (3.10), (3.11) and (3.13) one infers that

$$[\Delta_1\psi_1(s_i) \quad \Delta_2\psi_2(s_i)]\,\text{Adj}\,P(s_i)=0. \tag{3.14}$$

From (3.8), (3.9) and (3.14) it follows that $N_c(s), D_c(s) \in H^{m\times m}$. Moreover from (3.1), (3.8), it is clear that N_c, D_c satisfy (3.6). Also $N_c(s)$, $D_c(s)$ are coprime for otherwise there exists $s^* \in \mathbb{C}_u$ such that $[N_c(s^*), D_c(s^*)]$ has rank $< m$. This implies that for every $i=1,\cdots,r$,

$$\det[N_c(s)N_i(s)+D_c(s)D_i(s)] \tag{3.15}$$

vanishes at s^*. It follows that $\det \psi_i(s^*)=0$, which contradicts the hypothesis on $\psi_i(s)$, $i=1,\cdots,r$. Finally from (3.8) and (3.12) it follows that $D_c^{-1}N_c$ is proper. Q.E.D.

3.3. Proof of Theorem 3.2. From (3.4) one infers that

$$\Delta_1(s_j)\psi_1\mathcal{K}_{i_01}(s_j) = -\Delta_2(s_j)\psi_2\mathcal{K}_{i_02}(s_j) \tag{3.16}$$

for all $j=1,\cdots,t$. Thus

$$\text{sign}\,[\det \Delta_1(s_j)\times \det \Delta_2(s_j)] = \text{sign}\,[\det(\psi_1\mathcal{K}_{i_01}(s_j))\det(\psi_2\mathcal{K}_{i_02}(s_j))]. \tag{3.17}$$

The proof now follows from the observation that the sign of the left-hand side of (3.17) does not change for $j=1,\cdots,t$. Q.E.D.

4. Simultaneous partial pole placement of r min $(m,p)=1$ plants.

4.1. In this section we consider the simultaneous stabilization problem of r single input or single output systems. The case $r \leqq \max(m,p)$ has been considered in Theorem 2.5. Here we restrict attention to $r>\max(m,p)$. First of all we parametrize the set of all compensators simultaneously stabilizing a generic m tuple of $1\times m$ plants and prove the following.

THEOREM 4.1. *The problem of simultaneously stabilizing a* $m+p$ *tuple of* min $(m,p)=1$ *systems is equivalent to the problem of stabilizing a* min $(m,p)=1$ *system by a minimum phase compensator.*

In particular for $m=p=1$ we obtain the following corollary.

COROLLARY 4.2. *The following three problems are equivalent.*

(i) *The problem of stabilizing a pair of single input single output systems.*

(ii) *The problem of stabilizing a single input single output system by a stable compensator.*

(iii) *The problem of stabilizing a single input single output system by a minimum phase compensator.*

Of course the equivalence between (i) and (ii) was originally obtained by Saeks and Murray [26], Vidyasagar and Viswanadham [34].

The importance of the partial pole placement problem is further emphasized by the following.

THEOREM 4.3. *Assume* min $(m,p)=1$. *The problem of simultaneously stabilizing* $m+p+k$, *systems (assume* $k\geqq 1$*) chosen generically is equivalent to the problem of simultaneous partial pole placement of* k *systems by a stable minimum phase compensator.*

In particular for $m=p=k=1$ we obtain

COROLLARY 4.4. (Ghosh [14]). *The problem of simultaneously stabilizing* 3 *single input single output plants chosen generically is equivalent to the problem of partially pole placing one single input single output plants by a stable minimum phase compensator.*

With reference to Theorem 4.3, we remark that the simultaneous partial pole placement problem of a single input single output plant by a stable, minimum phase compensator is precisely the transcendental Problem 1.4 and has been analyzed by Ghosh [14]. In particular conditions have been obtained which are separately necessary and sufficient. These conditions can be analogously stated for the min $(m,p)=1$ case and we refer to [12] for details.

4.2. Parametrizing the set of all compensators simultaneously stabilizing a generic m-tuple of $1\times m$ systems. Let us represent an m-tuple of $1\times m$ plants as

$$g_i(s)=\left[\frac{x_{i,1}(s)}{x_{i,m+1}(s)}\ \frac{x_{i,2}(s)}{x_{i,m+1}(s)}\cdots\frac{x_{i,m}(s)}{x_{i,m+1}(s)}\right] \tag{4.1}$$

where $x_{i,j}(s)\in H$, $i=1,\cdots,m$; $j=1,\cdots,m+1$. Let us also represent a $m\times 1$ compensator as

$$k(s)=\left[\frac{y_1(s)}{y_{m+1}(s)}\ \frac{y_2(s)}{y_{m+1}(s)}\cdots\frac{y_m(s)}{y_{m+1}(s)}\right]^T \tag{4.2}$$

where $y_j(s) \in H$, $j = 1, \cdots, m+1$. To say that the above m-tuple of systems (4.1) is stabilizable by the compensator (4.2) is to say that there exists $\Delta_1(s), \cdots, \Delta_m(s) \in J$ such that

$$\sum_{j=1}^{m+1} x_{i,j}(s) y_j(s) = \Delta_i(s) \tag{4.3}$$

$i = 1, \cdots, m$. Let

$$\mathbf{y}^h(s) = [y_1^h(s) \cdots y_{m+1}^h(s)] \tag{4.4}$$

be the solution of the homogeneous equation

$$\sum_{j=1}^{m+1} x_{i,j}(s) y_j(s) = 0 \tag{4.5}$$

$i = 1, \cdots, m$ and

$$\mathbf{y}^{p_i} = [y_1^{p_i}(s), \cdots, y_{m+1}^{p_i}(s)] \tag{4.6}$$

be a particular solution of the equation.

$$\begin{bmatrix} x_{1,1}(s) & \cdots & x_{1,m+1}(s) \\ \vdots & \vdots & \vdots \\ x_{m,1}(s) & & x_{m,m+1}(s) \end{bmatrix} \cdot \begin{bmatrix} y_1(s) \\ \vdots \\ y_{m+1}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i\text{th spot.} \tag{4.7}$$

A complete solution of (4.3) is given by

$$\mathbf{y}(s) = \delta(s) \mathbf{y}^h(s) + \sum_{i=1}^{m} \Delta_i(s) \mathbf{y}^{p_i}(s) \tag{4.8}$$

where $\delta(s) \in H$, $\Delta_i(s) \in J$ for $i = 1, \cdots, m$. The set of all compensators simultaneously stabilizing the m tuple of $\min(m, p) = 1$ systems is given by

$$\left[\frac{\delta(s) y_1^h(s) + \sum_{i=1}^m \Delta_i(s) y_1^{p_i}(s)}{\delta(s) y_{m+1}^h(s) + \sum_{i=1}^m \Delta_i(s) y_{m+1}^{p_i}(s)}, \cdots, \frac{\delta(s) y_m^h(s) + \sum_{i=1}^m \Delta_i(s) y_m^{p_i}(s)}{\delta(s) y_{m+1}^h(s) + \sum_{i=1}^m \Delta_i(s) y_{m+1}^{p_i}(s)} \right] \tag{4.9}$$

for some $\delta(s) \in H$, $\Delta_i(s) \in J$, $i = 1, \cdots, m$.

4.3. Proof of Theorem 4.1. Consider a set of $m+1$ tuple of $1 \times m$ plants given by (4.1). The set of all compensators that would stabilize a m-tuple of $1 \times m$ plants is given by (4.9). In order that it also stabilizes the $(m+1)$th plant, we need to satisfy the equation

$$\sum_{j=1}^{m+1} x_{m+1,j}(s) \left[\delta(s) y_j^h(s) + \sum_{i=1}^{m} \Delta_i(s) y_j^{p_i}(s) \right] = \Delta_{m+1}(s) \tag{4.10}$$

for some $\Delta_1, \cdots, \Delta_{m+1} \in J$, $\delta(s) \in H$.

Equation (4.10) can be rewritten as

$$\delta(s) \left[\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^h(s) \right] + \sum_{i=1}^{m} \Delta_i(s) \left[\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^{p_i}(s) \right] = \Delta_{m+1}(s). \tag{4.11}$$

Thus if a $m+1$ tuple of plants is simultaneously stabilizable, the plant

$$\left[\frac{\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^{p_1}(s)}{\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^h(s)} \cdots \frac{\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^{p_m}(s)}{\sum_{j=1}^{m+1} x_{m+1,j}(s) y_j^h(s)} \right] \tag{4.12}$$

is stabilizable by the minimum phase compensator

$$\left[\frac{\Delta_1(s)}{\delta(s)}, \cdots, \frac{\Delta_m(s)}{\delta(s)}\right]. \tag{4.13}$$

Conversely the minimum phase compensator (4.13) also stabilizes the $1 \times m$ plants

$$[kl_1(s), 0, \cdots, 0], [0, kl_2(s), 0, \cdots, 0], \cdots, [0, 0, \cdots, 0, kl_m(s)] \tag{4.14}$$

for $l_i(s) \in J$ and where $k \in \mathbb{R}$ is to be chosen arbitrarily large. □

4.4. Proof of Theorem 4.3. First of all, we shall need the following lemma.

LEMMA 4.1. *The problem of simultaneously stabilizing a pair of* $\min(m, p) = 1$ *systems chosen generically by a minimum phase compensator is equivalent to the problem of partially pole placing a* $\min(m, p) = 1$ *system by a stable, minimum phase compensator.*

Proof. As before we consider a pair of $1 \times m$ systems given by (4.1). Assume that these two systems are stabilizable by a minimum phase compensator (4.2) where $y_i(s) \in J$, $i = 1, \cdots, m$ and $y_{m+1}(s) \in H$, and

$$\begin{aligned} x_{1,1}(s)y_1(s) + \cdots + x_{1,m+1}(s)y_{m+1}(s) &= \Delta_1, \\ x_{2,1}(s)y_1(s) + \cdots + x_{2,m+1}(s)y_{m+1}(s) &= \Delta_2. \end{aligned} \tag{4.15}$$

Assume $x_{1,m+1}(s) \not\equiv 0$, $x_{2,m+1}(s) \not\equiv 0$; we have

$$\begin{aligned} y_{m+1}(s) &= \frac{1}{x_{1,m+1}(s)}\Delta_1(s) - \sum_{j=1}^{m} \frac{x_{1,j}(s)}{x_{1,m+1}(s)} y_j(s) \\ &= \frac{1}{x_{2,m+1}(s)}\Delta_2(s) - \sum_{j=1}^{m} \frac{x_{2,j}(s)}{x_{2,m+1}(s)} y_j(s) \end{aligned} \tag{4.16}$$

or

$$\begin{aligned} x_{2,m+1}(s)\Delta_1(s) - \sum_{j=1}^{m} x_{2,m+1}(s)x_{1,j}(s)y_j(s) = x_{1,m+1}(s)\Delta_2(s) \\ - \sum_{j=1}^{m} x_{1,m+1}(s)x_{2,j}(s)y_j(s) \end{aligned} \tag{4.17}$$

or equivalently

$$x_{2,m+1}(s)\Delta_1(s) + \sum_{j=1}^{m} [x_{1,m+1}(s)x_{2,j}(s) - x_{2,m+1}(s)x_{1,j}(s)]y_j(s) = x_{1,m+1}(s)\Delta_2(s). \tag{4.18}$$

We now claim the following. To say that (4.15) is solvable is to say that (4.18) is solvable for some $\Delta_1(s)$, $\Delta_2(s)$, $y_j(s) \in J$, $j = 1, \cdots, m$. Since (4.18) is the algebraic condition for the partial pole placement of a $\min(m, p) = 1$ plant, we have the lemma. Conversely note that if (4.18) is solvable then there exists a $y_{m+1}(s) \in H$ defined by (4.16) which satisfies (4.15), provided there does not exist $s^* \in \mathbb{C}_u$ where $x_{1,m+1}$, $x_{2,m+1}$, $(x_{1,m+1}x_{2,j} - x_{2,m+1}x_{1,j})$, $j = 1, \cdots, m$ vanish. Q.E.D.

The proof of Theorem 4.3 now follows by the following argument. Let $g_1, \cdots, g_{m+p+k}$ be a $m+p+k$ tuple of $\min(m, p) = 1$ systems. Consider the following k, $m+p+1$ tuples of systems

$$(g_1, \cdots, g_{m+p}, g_{m+p+1}), \cdots, (g_1, \cdots, g_{m+p}, g_{m+p+j}).$$

By Theorem 4.1 and Lemma 4.1, stabilizability of each of the above $m+p+1$ tuple of plants is equivalent to the partial pole assignment of a $\min(m, p) = 1$ plant by a stable minimum phase compensator. Thus to say that all the k, $m+p+1$ tuples are stabilizable is to say that a k tuple of $\min(m, p)=1$ plants are partially pole assignable by a stable, minimum phase compensator. Q.E.D.

5. Examples. In this section we construct the following two examples:

5.1. *Example* 1. Assume $\mathbb{C}_u$ to be the closed right half of the complex plane. Consider the following triplet of single input single output plants of McMillan degrees 1.

$$\frac{s-7}{s-4.6}, \quad \frac{s-2}{2s-2.6}, \quad \frac{s-6}{4.8s-24.6}. \tag{5.1}$$

We claim that every pair of plants in the above triplet are simultaneously stabilizable. This may be trivially checked, using the necessary and sufficient condition due to Saeks and Murray [26]. On the other hand, the above triplet is simultaneously stabilizable iff there exists $\Delta_1(s)$, $\Delta_2(s)$, $\Delta_3(s) \in J$ such that

$$\begin{aligned} &\Delta_1(s)[2.8(s-4)(s-3)]/[(s-1)(s-9)] \\ &\quad +\Delta_2(s)[(-3.8s^2+47.6s-144.6)]/[(s-1)(s-9)]+\Delta_3(s) \equiv 0. \end{aligned} \tag{5.2}$$

A necessary condition that (6.2) is satisfied is that the vectors

$$\pm\left[\frac{2.8(s-4)(s-3)}{(s-1)(s-9)} \quad \frac{-3.8s^2+47.6s-144.6}{(s-1)(s-9)} \quad 1\right] \tag{5.3}$$

miss an orthant in $\mathbb{R}^3$ for all $s \in \mathbb{C}_u$. For $s=0, 2, 3.5, 6$ the above vector (5.3) may be computed as

$$\pm[3.73 \quad -16.066 \quad 1], \qquad \pm[-.8 \quad 9.2286 \quad 1],$$
$$\pm[.0501 \quad 1.7854 \quad 1], \qquad \pm[-1.12 \quad -.28 \quad 1],$$

respectively. Thus it follows that the triplet of plants in (5.1) is simultaneously unstabilizable.

Remark. The transcendental problem which arises in the simultaneous stabilization of a triplet of plants $x_i(s)/y_i(s)$, $i=1, 2, 3$ is given by (1.2). A necessary condition that (1.2) is solvable is that every pair in the above triplet is simultaneously stabilizable. If the triplet is chosen generically, then it is easy to show that: "every pair in the above triplet is simultaneously stabilizable iff the total number of zeros of $\eta_{ij}(s)$, $\eta_{ik}(s)$ in between every two consecutive zeros of $\eta_{jk}(s)$ in any connected component of $\mathbb{R}_u$ is even, for $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$, where $\eta_{ij} = x_i y_j - x_j y_i$." The three plants in (5.1) are an example of such a triplet. It is interesting however that even when the above interlacing condition is satisfied, if the zeros of η_{ij}, η_{ik}, η_{jk} are interlaced in a connected component of $\mathbb{R}_u$ as shown in Fig. 5.1, the equation (1.2) cannot be solved and the triplet under consideration is simultaneously unstabilizable.

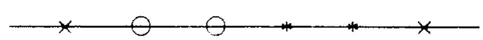

FIG. 5.1. *The zeros of η_{ij}, η_{jk}, η_{ik} in a connected component of $\mathbb{R}_u$ are denoted by "×", "○" and "*". The zero distribution represents an example of a triplet of plants that are not simultaneously stabilizable but are simultaneously stabilizable in pairs.*

Example 2. Consider the following triplet of 1×3 plants

$$\begin{bmatrix} \dfrac{-26}{18s-70} & \dfrac{54}{18s-70} & \dfrac{18s-42}{18s-70} \end{bmatrix}, \tag{5.4}$$

$$\begin{bmatrix} \dfrac{9s+10}{9s+2} & \dfrac{-9s}{9s+2} & \dfrac{9s+12}{9s+2} \end{bmatrix}, \tag{5.5}$$

$$\begin{bmatrix} \dfrac{-9s+9}{9s-9} & \dfrac{9s+9}{9s-9} & \dfrac{9s+9}{9s-9} \end{bmatrix}. \tag{5.6}$$

We now show that the above triplet is not simultaneously stabilizable.

Suppose not, and let (4.2) be the stabilizing compensator. Clearly we need to satisfy

$$\begin{bmatrix} \dfrac{-26}{s-a} & \dfrac{-54}{s-a} & \dfrac{-18s+42}{s-a} & \dfrac{-18s+42}{s-a} \\ \dfrac{9s+10}{s-a} & \dfrac{-9s}{s-a} & \dfrac{-9s}{s-a} & \dfrac{9s+2}{s-a} \\ \dfrac{-9s+9}{s-a} & \dfrac{9s+9}{s-a} & \dfrac{9s+9}{s-a} & \dfrac{9s-9}{s-a} \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \\ y_4(s) \end{bmatrix} = \begin{bmatrix} \Delta_1(s) \\ \Delta_2(s) \\ \Delta_3(s) \end{bmatrix} \tag{5.7}$$

for some $y_i(s) \in H$, $i=1,2,3,4$ and $\Delta_i(s) \in J$, $i=1,2,3$. In (5.7) "a" is chosen in such a way that $s-a$ vanishes in $\mathbb{C}_s$. It may be checked that the rows of the matrix in (5.7) are linearly dependent so that Δ_1, Δ_2, Δ_3 must satisfy

$$\Delta_1(s)[(s^2-1)/(s^2-s-6)]+\Delta_2(s)[(s^2-6s+8)/(s^2-s-6)]+\Delta_3(s)=0. \tag{5.8}$$

By evaluating the vector

$$[(s^2-1) \quad (s^2-6s+8) \quad (s^2-s-6)] \tag{5.9}$$

at $s=1.5, 2.5, 3.5, 4.5$ and using Lemma 2.1, we may infer that the triplet (5.4), (5.5), (5.6) is not simultaneously stabilizable.

Remark. Considering the fact that a generic triplet of 1×3 systems is simultaneously stabilizable, the triplet of plants in Example 2 is an "exceptional triplet." This serves to illustrate that the transcendental technique introduced in this paper can analyze "nongeneric" problems as well.

6. Conclusion. In this paper, we discuss in great details the application of scalar and matrix interpolation and transcendental methods in system design. In particular the transcendental problem arises in the partial pole assignment of a multiinput multioutput plant by a stable, minimum phase compensator. The latter problem seems to arise in the simultaneous stabilization problems and also in the nonswitching compensator problem [15]. The interpolation problem on the other hand is well known in control theory and has already been introduced in [26], [27], [34] and [35].

Among the results that we have obtained in this paper, the most significant result is that if $r \min(m, p) \leqq m+p$, the simultaneous partial pole assignment problem may be analyzed via interpolation methods and one obtains a semialgebraic parametrization of the partially pole assignable r-tuples of plants. On the other hand if $r \min(m, p) > m+p$ (as would be the case if $m=p$, $r \geqq 3$), the simultaneous partial pole assignment problem is to be analyzed via transcendental methods of the type introduced in this paper. The proposed transcendental approach, we hope, would become a new simultaneous system design methodology and may be generalized to include sensitivity minimization and other system design criterion as well.

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