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AN APPROACH TO SIMULTANEOUS SYSTEM DESIGN. PART II: NONSWITCHING GAIN AND DYNAMIC FEEDBACK COMPENSATION BY ALGEBRAIC GEOMETRIC METHODS*

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Abstract. This paper studies structured uncertainty problems in feedback system design, considers a compact parameterization of the space of linear dynamical systems and introduces "base points" and "critical points" as two algebraic-geometric objects that have significance in sensitivity and robustness studies, respectively. Using the Nevanlinna-Pick interpolation theory, the author obtains a necessary and sufficient condition for simultaneous stabilization of a structured one-parameter family of plants. A recent result due to Kharitonov, on the simultaneous stability of a parameterized family of polynomials, leads to a sufficiency condition for simultaneous stabilization of a structured multiparameter family of plants. Furthermore the author considers "simultaneous pole placement" of an r-tuple of plants as a means to arbitrarily tune the natural frequencies of a multimode linear dynamical system. The concept of "nondegenerate" and "twisted" r-tuples of plants is introduced as the pole placement problem is studied via Schubert enumerative geometry as an intersection problem on the associated Grassmannian. Various other design problems, viz., the strong stabilization problem and the dead beat control problem, are also considered.

Key words. simultaneous stabilization, pole assignment, base point, critical point, interpolation, nondegenerate, twisted

AMS(MOS) subject classifications. 93, 14

1. Introduction. In the last decade, control theorists have witnessed significant progress in multi-input multi-output system design. The central issue is the classification of plants or families of plants that admit a robust, nonswitching, dynamic output feedback compensator and satisfy a specific set of design constraints (viz. sensitivity minimization, stabilization, pole assignment, etc.). The basic problem without any additional design restriction is the robust stabilization problem described as follows.

Problem 1.1 (Nonswitching compensator problem). Given a family G of $p \times m$ real, linear dynamical systems, does there exist a nonswitching $m \times p$ real compensator $K(s)$ such that the closed-loop systems $G(s)[I+K(s)G(s)]^{-1}$ have poles only in the open left half of the complex plane for every $G(s) \in G$?

The above problem is important in the design of a compensator for a dynamical system whose parameters are uncertain and where Λ, the region of uncertainty, is known. This would indeed be the case if the parameters of the system were poorly identifiable. In this situation it is important to ascertain the existence of a compensator which is robust with respect to the parameter uncertainty and which stabilizes the family $(G_\lambda(s), \lambda \in \Lambda)$ of plants. Likewise, the above problem would arise if the dynamical system has a continuum of operating points (for example, the altitude of an aircraft or the r.p.m. of an induction motor). The parameters of the system may vary depending upon the choice of the operating points and the objective of the design is to synthesize a compensator robust with respect to the parameter variation. As has been originally pointed out by Saeks and Murray [37], this problem also arises if a control system $G(s)$ operates in many failed modes, all within a given family G, and the design

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objective is to continue to maintain stability even if a component of the control system $G(s)$ should fail. For a detailed discussion concerning many other illustrative examples of the above design problem, we refer the reader to the monograph edited by Ackermann [1], and many other references, e.g., [8]–[10].

We now detail the contribution of this paper. In § 2 we pose Problem 1.1 as a classification or parameterization problem and propose an algebraic geometric framework which enables us to characterize some of the design aspects of the stabilization and pole assignment problem that have not yet been recognized. In particular we compactify the space of dynamic compensators, study the continuity of the pole placement map, and show the existence of "base points." Finally we argue that the base points are significant, and that, in fact, around these points the location of the closed-loop poles is extremely sensitive with respect to the plant and the compensator parameters. We also study the asymptotic behavior of the closed-loop system as the plant/compensator parameters tend to the base points. We also show the existence of critical points, which are those infinite points in a neighborhood of which the stability of the closed-loop system is not robust with any margin of stability. The principal message of § 2 is therefore to avoid the base points and the critical points in any feedback system design.

In § 3 we describe explicit system design techniques for robust stabilization of a structured family of plants simultaneously. We consider the one-parameter family of plants and, using Nevanlinna–Pick interpolation methods [35], [36], obtain a necessary and sufficient condition for simultaneous stabilization of the family. We also obtain a bound on the degree of the compensator if it exists. Thus whereas the conjecture that "the set of pairs of simultaneously stabilizable plants of bounded degree has simultaneously stabilizing compensators of bounded degree" is false [13], in this paper we show the existence of a suitable class of one-parameter families of simultaneously stabilizable plants of bounded degree that admits simultaneously stabilizing compensators of bounded degree. In other words, the bound on the degree of the compensator can be a priori computed as a function of the degree of the plants to be compensated. In § 3 we also consider the multiparameter family of plants and obtain a sufficiency condition for simultaneous stabilization of such a family. Finally we consider robust simultaneous stabilization problems, wherein we study the extent to which a structured family of simultaneously stabilizable plants can be perturbed by an unstructured perturbation.

In §§ 4 and 5 we describe explicit system design techniques for pole assignment. In § 4 we consider a general ansatz for pole placement as a Schubert intersection problem [19] and derive explicit bounds on the degree of the compensator for generic simultaneous pole placement of proper and strictly proper plants. We also obtain similar bounds for the strong stabilization problem, wherein the dynamic compensator is assumed to be stable. In § 5 we analyze a simultaneous pole assignment problem via gain feedback. We also pose and analyze a classically well-known "dead beat control" problem.

Section 6 concludes this paper with a summary and discussion on future research directions.

2. A parameterization of the space of linear dynamical systems. In this section we consider the set of $m \times p$ proper or strictly proper transfer functions of McMillan degree q denoted, respectively, $S^q_{p,m}$ and $\Sigma^q_{p,m}$. In order to study degenerating families of systems and asymptotic properties of systems under deformation, it is useful to compactify the spaces $\Sigma^q_{p,m}$ and $S^q_{p,m}$ and obtain the spaces $\overline{\Sigma^q_{p,m}}$ and $\overline{S^q_{p,m}}$, respectively.

This construction enables us to study the behavior of a sequence of feedback control systems, in particular the closed-loop poles, as an asymptotic property of a sequence of compensators of a given McMillan degree q. Clearly the above study is useful in system identification, adaptive control and in high gain feedback control system design. It also enables us to study variations in the plant and in the compensator parameters simultaneously arising possibly as a result of parameter changes or structured uncertainty. (For some further details and previous work on families of systems, see [20]-[22], [25], and [32].) The main point that we wish to illustrate in this section is that in every feedback system design problem which involves a family of plants and a family of compensators, there exists a set of points called "base points" in the plant/compensator space which needs to be avoided. Otherwise near the base point the corresponding closed-loop poles are sensitive with respect to changes in the plant and the compensator parameters. Likewise we show that there exists a set of points called "critical points" in the compactified plant/compensator space which cannot be robustly stabilized by a nonswitching regulator.

2.1. A compactification of the space of dynamic compensators. Using a construction due to Hermann and Martin [26], compactification of $S^q_{p,m}$ has been developed by Brockett and Byrnes [3] for $q=0$ and more recently by Byrnes [4] for $q \geqq 0$. Although it is repetitive, we discuss the main results for the sake of completeness. (For a detailed description, see [12].)

To begin with, consider $S^0_{p,m}$ of $m\times p$ gain matrices $-K$. If we consider $-K$ to be a feedback gain matrix, it defines a relation between the m input u and p output y as follows:

$$[I_m \quad K][u^T \quad y^T]^T = 0. \tag{2.1.1}$$

The matrix $[I_m \ K]$ is an $m\times(m+p)$ matrix with linearly independent rows. In fact, multiplying (2.1.1) by a nonsingular $m\times m$ matrix does not change the relation between u and y. Thus the matrix $[I_m \ K]$ in fact defines an m-plane in $\mathbb{R}^{m+p}$ via its row span, and is therefore a point in Grass $(m, m+p)$. (See [19] for details about a Grassmannian.)

More generally, consider $S^q_{p,m}$ of $m\times p$ proper transfer functions $K(s)$ of McMillan degree q. If we consider $-K(s)$ to be a feedback compensator, it defines a relation between $u(s)$ and $y(s)$ as follows:

$$[I_m \quad K(s)][u(s)^T \quad y(s)^T]^T = 0. \tag{2.1.2}$$

Let us now consider the left coprime factorization of $K(s)$ given by

$$K(s) = \begin{bmatrix} k(s) & 0 \\ 0 & k(s)\end{bmatrix}^{-1} \begin{bmatrix} k_{11}(s) & k_{1p}(s) \\ k_{m1}(s) & k_{mp}(s)\end{bmatrix} \tag{2.1.3}$$

where $k(s)$, $k_{ij}(s)$, $i=1,\cdots,m$; $j=1,\cdots,p$ are polynomials in s of degree q. Let us write

$$k(s) \triangleq k^{(0)} + \cdots + k^{(q)}s^q, \tag{2.1.4}$$

$$k_{ij}(s) \triangleq k_{ij}^{(0)} + \cdots + k_{ij}^{(q)}s^q \tag{2.1.5}$$

and define

$$\underline{k} \triangleq [k^{(0)} \cdots k^{(q)}], \tag{2.1.6}$$

$$\underline{k}_{ij} \triangleq [k_{ij}^{(0)} \cdots k_{ij}^{(q)}], \tag{2.1.7}$$

$$\underline{u}_i \triangleq [u_i(s), su_i(s), \cdots, s^q u_i(s)]^T, \qquad i=1,\cdots,m, \tag{2.1.8}$$

$$\underline{y}_j \triangleq [y_j(s), sy_j(s), \cdots, s^q y_j(s)]^T, \qquad j=1,\cdots,p. \tag{2.1.9}$$

We now rewrite (2.1.2) as

$$\left[\begin{array}{cc|ccc} \underline{k} & 0 & \underline{k_{11}} & \cdots & \underline{k_{1p}} \\ & & \vdots & & \vdots \\ 0 & \underline{k} & \underline{k_{m1}} & \cdots & \underline{k_{mp}} \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_m \\ \underline{y_1} \\ \vdots \\ \underline{y_p} \end{bmatrix} = 0. \tag{2.1.10}$$

The associated matrix in (2.1.10) defines (via its row span) an m-plane in $\mathbb{R}^{(q+1)(m+p)}$ and is therefore a point in Grass $(m, (q+1)(m+p))$. Thus we have

$$S^q_{p,m} \subset \text{Grass}\,(m, (q+1)(m+p)). \tag{2.1.11}$$

Dually, by considering the right coprime factorization of $K(s)$ in (2.1.3), we also have

$$S^q_{p,m} \subset \text{Grass}\,(p, (q+1)(m+p)). \tag{2.1.12}$$

The closure $\overline{S^q_{p,m}}$ of $S^q_{p,m}$ inside either Grass $(m, (q+1)(m+p))$ or Grass $(p, (q+1)\times(m+p))$ is compact and is the compactification obtained by Byrnes [4] for $q \geqq 0$ and by Brockett and Byrnes [3] for $q=0$.

DEFINITION 2.1.1. The set of points

$$[\overline{S^q_{p,m}} - S^q_{p,m}]$$

is defined to be the "infinite points" in the above compactification.

Note that the infinite points are the set of limit points which needs to be added to $S^q_{p,m}$ in order to obtain a compact set.

The following example taken from [12] explicitly describes $\overline{S^1_{2,2}}$ in Grass (2, 8).

Example 2.1.2. Consider the set of two-input and two-output proper systems of degree 1 given by

$$G(s) = \begin{bmatrix} (a_7 s + a_1)/(a_6 s + a_5) & (a_8 s + a_2)/(a_6 s + a_5) \\ (a_9 s + a_3)/(a_6 s + a_5) & (a_{10} s + a_4)/(a_6 s + a_5) \end{bmatrix} \tag{2.1.13}$$

where

$$(a_7 a_5 - a_1 a_6)(a_8 a_5 - a_2 a_6) = (a_9 a_5 - a_3 a_6)(a_{10} a_5 - a_4 a_6). \tag{2.1.14}$$

The transfer function (2.1.13) defines the point

$$\text{row span of} \begin{bmatrix} a_5 & a_6 & 0 & 0 & a_1 & a_7 & a_2 & a_8 \\ 0 & 0 & a_5 & a_6 & a_3 & a_9 & a_4 & a_{10} \end{bmatrix} \tag{2.1.15}$$

in Grass (2, 8). Not every element in Grass (2, 8), however, corresponds to an element in $S^1_{2,2}$ or in $\overline{S^1_{2,2}}$. To see that, consider the point p defined by

$$p \triangleq \text{row span of} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{bmatrix} \tag{2.1.16}$$

in Grass (2, 8). In order that $p \in S^1_{2,2}$ it should satisfy

$$\eta_{12} = \eta_{34} = 0, \tag{2.1.17}$$

$$\eta_{14} = \eta_{23}, \tag{2.1.18}$$

$$\eta_{13}\eta_{68} + \eta_{24}\eta_{57} + \eta_{23}[\eta_{58} + \eta_{67}] = 0, \tag{2.1.19}$$

$$\eta_{24} \neq 0, \tag{2.1.20}$$

and

(2.1.21) $(\eta_{24}\eta_{54} \neq \eta_{14}\eta_{64}$ or $\eta_{24}\eta_{74} \neq \eta_{14}\eta_{84}$ or $\eta_{24}\eta_{25} \neq \eta_{14}\eta_{26}$ or $\eta_{24}\eta_{27} \neq \eta_{14}\eta_{28})$

where $\eta_{ij} = x_i y_j - x_j y_i$ is the set of Plucker coordinates (see [7]) via which Grass (2, 8) can be imbedded in $\mathbb{RP}^{27}$. Note that the infinite points in $\overline{S^1_{2,2}}$ are described by (2.1.17)-(2.1.19) and

(2.1.22) $$\eta_{24} = 0$$

or

(2.1.23) $$\frac{\eta_{24}}{\eta_{14}} = \frac{\eta_{64}}{\eta_{54}} = \frac{\eta_{84}}{\eta_{74}} = \frac{\eta_{26}}{\eta_{25}} = \frac{\eta_{28}}{\eta_{27}}.$$

We remark that (2.1.17)-(2.1.19) and (2.1.22) contain the set of improper transfer functions in $\overline{S^1_{2,2}}$, and that (2.1.17)-(2.1.19) and (2.1.23) describe the set of transfer functions of degree 0 in $\overline{S^1_{2,2}}$.

By analogous technique we can compactify $\Sigma^q_{p,m}$, the set of $m \times p$ strictly proper transfer functions of degree q as a point in Grass $(m, (q+1)m+qp)$ or Grass $(p, (q+1)p+qm)$. Let us now consider the following example.

Example 2.1.3. Consider the set $\Sigma^1_{2,2}$ given by (2.1.13), (2.1.14) with $a_7 = a_8 = a_9 = a_{10} = 0$. Every transfer function in $\Sigma^1_{2,2}$ defines a point in Grass (2, 6) given by

(2.1.24) $$\text{row span of} \begin{bmatrix} a_5 & a_6 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_5 & a_6 & a_3 & a_4 \end{bmatrix}.$$

In fact if p is a point in Grass (2, 6) defined by

(2.1.25) $$\text{row span of} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix},$$

then $p \in \Sigma^1_{2,2}$ if and only if

(2.1.26) $$\eta_{12} = \eta_{34} = 0, \qquad \eta_{14} = \eta_{23}, \qquad \eta_{56} = 0, \qquad \eta_{24} \neq 0.$$

It follows that $\overline{\Sigma^1_{2,2}} - \Sigma^1_{2,2}$ is described by

(2.1.27) $$\eta_{12} = \eta_{34} = \eta_{56} = \eta_{24} = 0, \qquad \eta_{14} = \eta_{23} = 0.$$

2.2. Continuity of the pole placement map and the base locus condition. Classically, root locus [11] was studied by considering a single-input single-output plant together with a scalar gain feedback, and a plot of the closed-loop poles as a function of the feedback gain was obtained. Since root locus is continuous with respect to the scalar gain, the closed-loop poles asymptotically and continuously approach the open-loop zeros as the gain tends to infinity. For the multi-input multi-ouput case the generalization is not immediate. However, as we shall see in this section, the asymptotic locus of the closed-loop poles as a function of the compensator parameters is not necessarily continuous. Furthermore, as we have seen in § 2.1, infinity is not a single point and therefore there is more than one way to tend to infinity leading to distinct closed-loop behavior of the dynamical system. These are now detailed as follows.

Let $G(s)$ be a proper $p \times m$ plant of McMillan degree n and let $K(s)$ be a proper $m \times p$ compensator of McMillan degree q. Let us represent them in their coprime fraction representation as follows:

(2.2.1) $$G(s) = N_p(s) D_p(s)^{-1},$$

(2.2.2) $$K(s) = D_c(s)^{-1} N_c(s)$$

where $N_p(s)$, $D_p(s)$, $N_c(s)$, $D_c(s)$ are matrices of appropriate orders with elements in the ring H of stable proper rational functions. It is well known that $K(s)$ places the poles of the plant $G(s)$ at $s = s_1, s_2, \cdots, s_{n+q}$ provided

(2.2.3) $$\det[N_c(s)N_p(s) + D_c(s)D_p(s)](s_i) = 0$$

for $i = 1, \cdots, n+q$. Let

(2.2.4) $$c_0 + c_1 s + \cdots + c_{n+q}s^{n+q} \triangleq p(s)$$

be a polynomial which vanishes precisely where $\det[N_cN_p + D_cD_p]$ vanishes. The polynomial $p(s)$ is called the characteristic polynomial. For a fixed $G(s)$, (2.2.3) defines a mapping between the compensator parameters and the coefficients c_i of the characteristic polynomial $p(s)$ in (2.2.4). We therefore define the pole placement map

(2.2.5) $$\chi: \quad \overline{S^q_{p,m}} - B \to \mathbb{RP}^{n+q}$$

defined by

(2.2.6) $$\chi(K(s)) = [c_0, \cdots, c_{n+q}]$$

where the right-hand side of (2.2.6) is written in homogeneous coordinates. The set B is called the base locus (see [19], [34]) and is defined to be the set of points in $\overline{S^q_{p,m}}$, where χ does not have a continuous extension. It is a well-known fact (see [34, p. 98]) that the base locus B is given by

(2.2.7) $$B \triangleq \bigcap_H \overline{\chi^{-1}(H)}$$

where H is a hyperplane in $\mathbb{RP}^{n+q}$ and $\overline{\chi^{-1}(H)}$ is the closure of $\chi^{-1}(H)$ inside $\overline{S^q_{p,m}}$. The significance of the base locus is made clear in the following proposition.

PROPOSITION 2.2.1. *The base locus B is equivalently described as*

(2.2.8) $$B = \{K(s) \mid \det[N_c(s)N_p(s) + D_c(s)D_p(s)] \equiv 0\}.$$

Proof. Every $s_i \in \mathbb{R}$ defines a hyperplane H_i in $\mathbb{RP}^{n+q}$, viz., the hyperplane orthogonal to the vector $[1 \quad s_i \quad s_i^2 \cdots s_i^{n+q}]$. It follows that

(2.2.9) $$\overline{\chi^{-1}(H_i)} = \{K(s) \mid \det[N_cN_p + D_cD_p](s_i) = 0\}.$$

Let $s_1, \cdots, s_{n+q+1}$ be a set of distinct real numbers. It follows that

(2.2.10) $$\bigcap_{i=1}^{n+q+1} \overline{\chi^{-1}(H_i)} = \{K(s) \mid \det[N_cN_p + D_cD_p](s) \equiv 0\},$$

since the characteristic polynomial of the closed-loop transfer function is of degree $n+q$. Finally since

(2.2.11) $$B \subset \bigcap_{i=1}^{n+q+1} \overline{\chi^{-1}(H_i)},$$

we have

(2.2.12) $$B \subset \{K(s) \mid \det[N_c(s)N_p(s) + D_c(s)D_p(s)] \equiv 0\}.$$

Conversely, let us assume that

(2.2.13) $$K_1(s) \in \{K(s) \mid \det[N_c(s)N_p(s) + D_c(s)D_p(s)] \equiv 0\}.$$

Clearly $K_1(s)$ is a compensator for which the associated characteristic polynomial $p(s)$ is identically zero, i.e., $c_0 = c_1 = \cdots = c_{n+q} = 0$. Hence χ cannot be defined at $K_1(s)$. By choosing two different sequences of compensators approaching $K_1(s)$, we can show that χ does not even have a continuous extension to a mapping which is defined at $K_1(s)$. Otherwise χ has to be multivalued at $K_1(s)$, which is absurd. Therefore

$$\{K(s) \mid \det [N_c(s)N_p(s) + D_c(s)D_p(s)] = 0\} \subset B. \tag{2.2.14}$$ □

In the following example we characterize the base locus for a particular compensation problem.

Example 2.2.2. Consider a single-input single-output plant $g(s)$ of degree 1 given by $(s+1)/(s+2)$. Consider a proper compensator $k(s)$ of degree 1 given by $(k_1 s - 2)/(s+k_2)$. The compensator is parameterized by two parameters k_1 and k_2. The closed-loop transfer function is given by

$$\frac{(s+1)(s+k_2)}{(k_1+1)s^2 + (k_1+k_2)s + (2k_2-2)}. \tag{2.2.15}$$

Using Proposition 2.2.1 we infer that the base locus B is the point set given by

$$B \triangleq \{k(s) \mid k_1 = -1,\ k_2 = 1\}. \tag{2.2.16}$$

The pole placement map χ is described as

$$\chi\colon\ S^1_{1,1} - B \to \mathbb{RP}^2, \tag{2.2.17}$$

$$\chi(k(s)) = \{k_1 + 1,\ k_1 + k_2,\ 2k_2 - 2\}. \tag{2.2.18}$$

Clearly χ is continuous everywhere except at $k(s) = -(s+2)/(s+1)$ where it is not even defined. To show that χ cannot be extended to a function χ_1 continuous at $-(s+2)/(s+1)$, we assume the parameterized family of compensators F_t given by

$$F_t \triangleq \{k(s) \mid (k_1+1) = t(k_2-1)\} \tag{2.2.19}$$

where $t \in \mathbb{R}$. Consider a sequence of compensators $h_1^{(t)}(s), h_2^{(t)}(s), \cdots$ in F_t approaching the compensator $-(s+2)/(s+1)$. It is easy to see that

$$\chi(h_i^{(t)}(s)) = [t \quad (t+1) \quad 2] \tag{2.2.20}$$

independent of i but dependent on t. Note that the closed-loop system is stable if $t > 0$ and unstable otherwise. This indicates that when the compensator parameters k_1, k_2 approach arbitrarily close to the point $(-1, 1)$, possibly along two different lines in the (k_1, k_2) plane, the closed-loop system can be stable in one case and unstable in the other. Thus stability is not a robust property in the neighborhood of $(-1, 1)$ primarily because the closed-loop poles are extremely sensitive with respect to the parameter t. Finally the asymptotic locus of the roots of the characteristic polynomial is not continuous with respect to the parameters k_1, k_2 at the point $(-1, 1)$. Hence χ does not have a continuous extension to a function χ_1 defined at the point $(-1, 1)$.

2.3. Degeneration of a family of closed-loop systems. In the last section we derived the base locus condition assuming that the plant is fixed and the compensator parameters are allowed to vary. In this section we describe a general theory, wherein both the plant and the compensator parameters are allowed to vary. Let us consider the set $\overline{S^n_{m,p}}$ of $p \times m$ plants of degree n and the set $\overline{S^q_{p,m}}$ of $m \times p$ compensators of degree q suitably compactified. As an extension of the pole placement map (2.2.5) we define

$$\psi\colon\ \overline{S^n_{m,p}} \times \overline{S^q_{p,m}} - \tilde{B} \to \mathbb{RP}^{n+q}, \tag{2.3.1}$$

$$\psi(G(s), K(s)) = [c_0, \cdots, c_{n+q}] \tag{2.3.2}$$

where $G(s)$, $K(s)$, c_i, $i=0,\cdots,n+q$ are described via (2.2.1)–(2.2.4). Similar to the definition of B, we define $\tilde{B}$ to be the base locus of the map ψ. We now consider the following proposition.

PROPOSITION 2.3.1. *The base locus $\tilde{B}$ is described as*

$$\tilde{B}=\{(G(s),K(s))\,|\,\underline{\sigma}[N_c(s)N_p(s)+D_c(s)D_p(s)]=0\ \forall s\in\mathbb{C} \quad \text{where } \underline{\sigma}(\cdot) \text{ denotes the minimum singular value}\}. \tag{2.3.3}$$

Proof. The proof follows from Proposition 2.2.1 since

$$\underline{\sigma}[N_c(s)N_p(s)+D_c(s)+D_p(s)]=0 \quad \text{for all } s\in\mathbb{C}, \tag{2.3.4}$$

$$\Leftrightarrow \det[N_c(s)N_p(s)+D_c(s)D_p(s)]\equiv 0. \tag{2.3.5}$$

□

We now consider the following illustrative example from the literature [33, Ex. 2.1], wherein the compensator is fixed and the plant parameters degenerate into the base locus.

Example 2.3.2. Assume $b_{12}\neq 0$. Consider the following state-space system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & b_{12}\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}, \tag{2.3.6}$$

$$[y_1^T \quad y_2^T]^T=[x_1^T \quad x_2^T]^T \tag{2.3.7}$$

together with the feedback compensator

$$[u_1^T \quad u_2^T]^T=-[y_1^T \quad y_2^T]^T. \tag{2.3.8}$$

The closed-loop system above is stable and has a pair of poles at -2, -2. Let us now perturb (2.3.6) and consider

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & b_{12}\\ 5/b_{12} & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}. \tag{2.3.9}$$

The perturbed system (2.3.9), (2.3.7), (2.3.8) is an unstable system with closed-loop poles at the zeros of $(s+2)^2-5$. Note that the stability of the closed-loop system and the instability of the perturbed closed-loop system are independent of b_{12}. This fact is quite disturbing because, for arbitrary large b_{12}, the matrix components of the two systems are "close."

From the viewpoint of the parameterization proposed in § 2.1, it may be trivially checked that the unperturbed system defines a point in Grass (2, 8) given by

$$\text{row span of}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & b_{12} & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\end{bmatrix} \tag{2.3.10}$$

which degenerates to the point

$$\text{row span of}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\end{bmatrix} \tag{2.3.11}$$

as $b_{12}\to\infty$. The perturbed system, on the other hand, defines the point

$$\text{row span of}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & b_{12} & 0\\ 0 & 0 & b_{12} & b_{12} & 5 & 0 & 1 & 0\end{bmatrix} \tag{2.3.12}$$

which also degenerates to the point (2.3.11) as $b_{12}\to\infty$. Thus the two closed-loop systems under consideration are "close" as points on Grass (2, 8).

It may be trivially checked that the point (2.3.11) is on the base locus of the map

$$\psi:\quad \overline{S^2_{2,2}} \times \overline{S^0_{2,2}} - \tilde{B} \to \mathbb{RP}^2. \tag{2.3.13}$$

Thus, following the argument presented in Example 2.2.2, we conclude that stability is not a robust property around the point (2.3.11). This may also be viewed as an algebraic-geometric interpretation of the conclusion presented in Example 2.1 of [33].

Remark 2.3.3. Proposition 2.3.1 serves to justify the comment on page 78 of [33]: "A multivariable system will not be robust ... if its return difference transfer function matrix ... is nearly singular at some frequency ...." In fact, a near singular transfer function matrix is close to the base locus.

In multivariable feedback system design, a good design strategy is to avoid the base locus. In particular, around the base locus, the closed-loop poles are sensitive with respect to the plant and the compensator parameters. (For an original reference on the application of base locus in system theory, see Byrnes [45].)

2.4. Robust system design and the critical point condition. Let $\overline{S^q_{p,m}}$ be as defined in § 2.1. For the purpose of robust system design, in this section we pose the following problem.

Problem 2.4.1. Parameterize the set $\Omega^q_{p,m}$ of points in $\overline{S^q_{p,m}}$ with the property that for every $\sigma \in \Omega^q_{p,m}$ there exists an open neighborhood $N(\sigma)$ of σ in $\overline{S^q_{p,m}}$ such that every plant in $S^q_{p,m} \cap N(\sigma)$ is simultaneously stabilizable.

Analysis of the above problem is clearly important in the study of "system degeneration" and "high gain" compensation. (For motivational remarks and other details pertaining to the case $m = p = 1$, see [6].)

DEFINITION 2.4.2. The set of points

$$\overline{S^q_{p,m}} - \Omega^q_{p,m} \tag{2.4.1}$$

is defined as the "critical points" in the compactified $\overline{S^q_{p,m}}$.

We now state the following simple proposition.

PROPOSITION 2.4.3. *Every critical point is an infinite point in* $\overline{S^q_{p,m}}$.

Proof. The finite points in $\overline{S^q_{p,m}}$ are points in $S^q_{p,m}$. Moreover, every point in $S^q_{p,m}$ has an open neighborhood in $S^q_{p,m}$ that can be simultaneously stabilized by a dynamic compensator. Therefore no point in $S^q_{p,m}$ is a critical point. □

The set of critical points are now explicitly characterized as follows.

THEOREM 2.4.4. *The points*

$$\text{row span of} \begin{bmatrix} \underline{k} & & 0 & \underline{k_{11}} & \underline{k_{12}} & \cdots & \underline{k_{1p}} \\ & \ddots & & \vdots & \vdots & & \vdots \\ 0 & & \underline{k} & \underline{k_{m1}} & \underline{k_{m2}} & \cdots & \underline{k_{mp}} \end{bmatrix} \tag{2.4.2}$$

belonging to $\overline{S^q_{p,m}}$ *are critical points of* $\overline{S^q_{p,m}}$ *if and only if there exists* s^* *in the closed right half of the complex plane such that the row rank of the matrix* $M(s)$ *given by*

$$M(s) \triangleq \begin{bmatrix} k(s) & & 0 & k_{11}(s) & \cdots & k_{1p}(s) \\ & \ddots & & \vdots & & \vdots \\ 0 & & k(s) & k_{m1}(s) & \cdots & k_{mp}(s) \end{bmatrix} \tag{2.4.3}$$

at $s = s^*$ *is* $< m$.

Remark 2.4.5. The vectors $\underline{k}$, $\underline{k}_{ij}$ and the polynomials $k(s)$, $k_{ij}(s)$, $i = 1, \cdots, m$; $j = 1, \cdots, p$ are defined in (2.1.4)-(2.1.7).

Proof of Theorem 2.4.4. The sufficiency is clear, since the closed-loop characteristic polynomial

$$\det\left[\begin{array}{ccc|ccc} k(s) & & 0 & k_{11}(s) & \cdots & k_{1p}(s) \\ & \ddots & & \vdots & & \vdots \\ 0 & & k(s) & k_{m1}(s) & \cdots & k_{mp}(s) \\ \hline & K(s) & & & I & \end{array}\right] \tag{2.4.4}$$

corresponding to any $p\times m$ compensator transfer function $K(s)$ always vanishes at s^* and therefore (2.4.2) is indeed a critical point.

In order to show "necessity" assume that the row rank of the matrix $M(s)$ in (2.4.3) is m for all $s\in\mathbb{C}$. Then clearly $M(s)$ is right invertible, i.e., there exists an $(m+p)\times m$ matrix $N(s)\,|\,M(s)N(s)=\Delta(s)$, where $\det\Delta(s)$ vanishes only in $\mathbb{C}_s$, the open left half of the complex plane. We partition $N(s)$ as $[N_1(s)^T\ N_2(s)^T]^T$, where $N_1(s)$ is $m\times m$ and where we choose $N(s)$ such that $\det N_1(s)\not\equiv 0$ and $N_1^{-1}N_2$ is proper. Thus we have a transfer function $N_1^{-1}N_2$ of a compensator which stabilizes the transfer function corresponding to the plant (2.4.2). Moreover, a sufficiently small neighborhood of (2.4.2) in $S^q_{p,m}$ is stabilized by the compensator. It follows that (2.4.2) is not a critical point. The only case left is that where the rank of the matrix $M(s)$ in (2.4.3) is $<m$ at some points in $\mathbb{C}_s$, the open left half of the complex plane. This fact, however, implies that $M(s)$ can be factored as

$$M(s)=\Delta_1(s)M_1(s) \tag{2.4.5}$$

where $\det\Delta_1(s)$ vanishes at $s=s_1$ and where $M_1(s)$ has either rank m at all $s\in\mathbb{C}_s$ or rank $<m$ for some $s=s_2$ in $\mathbb{C}_s$. Proceeding as above it follows that $M(s)$ can be factored as

$$M(s)=\Delta^*(s)M^*(s) \tag{2.4.6}$$

where $\det\Delta^*(s)$ vanishes only in $\mathbb{C}_s$ and where $M^*(s)$ is of rank m at all $s\in\mathbb{C}$. By obtaining a right inversion of $M^*(s)$ and considering the reasoning as before it follows that (2.4.2) is not a critical point. □

Example 2.4.6. If $q=m=p=1$, the space $S^1_{1,1}$ contains proper transfer functions

$$q(s)=\frac{a_1s+a_0}{s+b_0},\qquad a_0\neq a_1b_0 \tag{2.4.7}$$

of degree 1, that can be parameterized as points in $\mathbb{RP}^3$ via the homogeneous coordinates $[1\ \ b_0\ \ a_1\ \ a_0]$. The set of infinite points $\overline{S^1_{1,1}}-S^1_{1,1}$ in $\mathbb{RP}^3$ is described as

$$\{[h_0,h_1,h_2,h_3]\colon h_0h_3=h_1h_2\} \tag{2.4.8}$$

and the set of critical points are described as

$$\{[h_0,h_1,h_2,h_3]\colon h_0h_3=h_1h_2 \text{ and } h_0h_1\leqq 0 \text{ and } h_2h_3\leqq 0\}. \tag{2.4.9}$$

Remark 2.4.7. The significance of a critical point may now be emphasized. By definition, if ξ is a critical point in $\overline{S^q_{p,m}}$, then there exists a sequence of points ξ_i, $i=1,2,\cdots$, in $S^q_{p,m}$ which tends to ξ in the limit and with the property that the family of plants $\{\xi_i\}$ is simultaneously unstabilizable by a dynamic compensator. Since critical points are those infinite points in $\overline{S^q_{p,m}}$, neighborhoods of which are not simultaneously stabilizable, a good design strategy is to avoid the critical points in system design.

In order to describe Corollary 2.4.9 of Theorem 2.4.4 we need the following definition.

DEFINITION 2.4.8. A compensator is said to stabilize a plant with margin of stability $\varepsilon > 0$ if the closed-loop poles are in the region $\{s \in \mathbb{C}: \text{Re } s < -\varepsilon\}$.

COROLLARY 2.4.9. *Let $\xi_1(s), \xi_2(s), \cdots$ be a sequence of plants in $S^q_{p,m}$ which approaches a critical point in $\overline{S^q_{p,m}} - \Omega^q_{p,m}$ arbitrarily close. Then the family $\{\xi_i(s), i = 1, 2, \cdots\}$ of plants is not simultaneously stabilizable with any margin of stability.*

Proof. Suppose the corollary does not hold. Then there exists a compensator which simultaneously stabilizes the given family with a margin of stability $\varepsilon > 0$. This, however, implies that the plants $\xi_i(s)$ in the family are not in the associated base locus. Thus the closed-loop poles are continuous with respect to the plant parameters. It follows from the definition of a critical point that there exists an integer N sufficiently large that $\xi_N(s)$ has a closed-loop pole arbitrarily close to a point s^* in the closed right half of the complex plane. However, this is a contradiction. □

3. Simultaneous stabilization of plants with structured uncertainty. In this section we restrict our attention to single-input single-output plants and consider the following problem.

Problem 3.1. Let P be a subset of $\Omega^n_{1,1}$ for some integer n. Does there exist a dynamic compensator of degree $\leqq q$ which stabilizes each and every plant $\sigma \in P$?

Of course if σ_0 is a plant in P and if the subset P represents structured perturbation of σ_0 in $\Omega^n_{1,1}$, the above Problem 3.1 is a robust stabilization problem with structured uncertainty. Also, for the purpose of estimating the complexity of the compensator which solves a particular design problem, it is of interest to consider.

Question 3.2. Is q a priori bounded?

If P is a single point it is well known that an upper bound on q is given by $q \leqq n$. On the other hand it has been shown [13], [12] that if P is a pair of points in $\Omega^n_{1,1}$, an upper bound on q does not exist.

3.1. A general ansatz for simultaneous stabilization. In this section, we consider a family P of single-input single-output proper plants of degree n given by

$$P = \{x_\lambda(s)/y_\lambda(s): \lambda \in \Lambda, x_\lambda, y_\lambda \in H; x_\lambda, y_\lambda \text{ are coprime and } \deg x_\lambda/y_\lambda = n \text{ for all } \lambda \in \Lambda\}. \tag{3.1.1}$$

Assume that P contains at least two distinct plants. Furthermore let us define

$$\eta_{ij}(s) = x_i(s)y_j(s) - x_j(s)y_i(s), \qquad i, j \in \Lambda \tag{3.1.2}$$

where Λ is an arbitrary parameter space. We now state without proof the following result from Theorem 5.1 of [13].

PROPOSITION 3.1.1. *Let $x_1(s)/y_1(s)$, $x_2(s)/y_2(s)$ be two distinct plants in P. There exists a proper compensator which simultaneously stabilizes each and every plant in P if and only if there exist $\Delta_1(s), \Delta_2(s) \in J$, the set of multiplicative units of H, such that*

(i) *$\Delta_1 y_2 - \Delta_2 y_1$ and $\Delta_2 x_1 - \Delta_1 x_2$ vanish at $s_1, \cdots, s_t \in \mathbb{C}_u$ with multiplicities at least $m_1, \cdots, m_t$, respectively, where $s_1, \cdots, s_t$ are the zeros of $\eta_{12}(s)$ in $\mathbb{C}_u$ with multiplicities $m_1, \cdots, m_t$, respectively. (Here $\mathbb{C}_u = \mathbb{C} - \mathbb{C}_s$.)*

(ii) *$\Delta_2 x_1 - \Delta_1 x_2$ does not vanish at ∞ with multiplicity m_∞ unless $\eta_{12}(s)$ vanishes at ∞ with multiplicity at least m_∞.*

(iii) *There exists $\Delta_\lambda \in J$ for all $\lambda \in \Lambda - \{1, 2\}$ such that*

$$\Delta_1 \eta_{\lambda 2} - \Delta_2 \eta_{\lambda 1} = \Delta_\lambda \eta_{12}. \tag{3.1.3}$$

In the proof of the next theorem we will introduce four auxiliary return difference polynomials which we denote by $\alpha_i(s)$, $i = 1, 2, 3, 4$.

We now consider the main result of this section described in the following theorem.

THEOREM 3.1.2. *Assume that P contains at least two proper plants and does not contain three plants* g_1, g_2, g_3 *such that* $g_1(s^*) = g_2(s^*) = g_3(s^*)$ *for any* $s^* \in \mathbb{C} \cup \{\infty\}$. *Furthermore assume that* Λ *is a compact subset of* $\mathbb{R}^\alpha$ *for some integer* α *and that the parameters of* $x_\lambda(s)/y_\lambda(s)$ *depend continuously on* λ. *A sufficient condition that every plant in P is simultaneously stabilizable by a dynamic compensator is given by the stability of the four polynomials* $\alpha_i(s)$, $i = 1, 2, 3, 4$, *defined in* (3.1.10).

Remark 3.1.3. For details on the simultaneous stabilization of a finite number of plants, see [12]-[14], [16], [17], [37], [42].

Remark 3.1.4. The above problem is a version of the robust stabilization problem. It is indeed surprising that such a problem may be tackled via simultaneous stabilization of a finite number of plants, in particular, four. We consider Theorem 3.1.2 to be a theoretically significant result, since simultaneous stabilization of four plants by a stable, minimum phase compensator (currently an open problem) is conceptually simpler than simultaneous stabilization of a family of uncountably many plants.

Proof of Theorem 3.1.2. Let us consider the two proper plants $g_1(s), g_2(s)$ in P defined as follows:

$$g_1(s) = x_1(s)/y_1(s), \qquad g_2(s) = x_2(s)/y_2(s)$$

where $x_1, x_2, y_1, y_2 \in H$ and x_i, y_i are coprime $i = 1, 2$. Assume that there exists $\Delta_1, \Delta_2, \Delta_\lambda \in J$, $\lambda \in \Lambda - \{1, 2\}$ such that (3.1.3) is satisfied. We now show that P is simultaneously stabilizable by a possibly improper compensator.

By easy algebraic manipulation of (3.1.3) we first obtain the following two identities:

$$[\Delta_1 x_2 - \Delta_2 x_1]\eta_{\lambda 1} = [\Delta_\lambda x_1 - \Delta_1 x_\lambda]\eta_{12}, \tag{3.1.4}$$

$$[\Delta_1 y_2 - \Delta_2 y_1]\eta_{\lambda 1} = [\Delta_\lambda y_1 - \Delta_1 y_\lambda]\eta_{12}. \tag{3.1.5}$$

Noting that $\eta_{\lambda 1}(s)$ does not have a zero at any point where $\eta_{12}(s)$ has a zero for all $\lambda \in \Lambda - \{1, 2\}$, we find from (3.1.4) and (3.1.5) that condition (i) of Proposition 3.1.1 is satisfied. Since condition (ii) of Proposition 3.1.1 is imposed in order to make the compensator proper, it follows that P is simultaneously stabilizable by a possibly improper compensator.

Since Λ is compact we now claim that P is actually stabilizable by a proper compensator. This is because there exists a $\sigma_1, \sigma_2 > 0$ such that the closed-loop poles with respect to the improper compensator is in the region $\{s: -\sigma_2 < \operatorname{Re} s < \sigma_1\}$. It follows that we can perturb the improper compensator and obtain the required proper compensator.

From the argument presented so far we conclude that a necessary and sufficient condition for the simultaneous stabilization of P is the existence of $\Delta_1, \Delta_2, \Delta_\lambda \in J$, $\lambda \in \Lambda - \{1, 2\}$ such that (3.1.3) is satisfied.

We now obtain a sufficiency condition for the existence of $\Delta_1, \Delta_2, \Delta_\lambda \in J$, $\lambda \in \Lambda - \{1, 2\}$ such that (3.1.3) is satisfied.

Clearing the denominator, we may assume without any loss of generality that in (3.1.3), $\Delta_1, \Delta_2, \Delta_\lambda, \eta_{\lambda 2}, \eta_{\lambda 1}$ and η_{12} are polynomials of appropriate degrees. By a suitable algebraic manipulation we conclude the existence of two polynomials $\eta'_{\lambda 2}(s)$ and $\eta'_{\lambda 1}(s)$ such that the following two statements are equivalent:

(1) There exist $\Delta_1, \Delta_2, \Delta_\lambda \in J$, $\lambda \in \Lambda - \{1, 2\}$ such that (3.1.3) is satisfied.

(2) There exist stable polynomials Δ'_1, Δ'_2 such that $\Delta'_1 \eta'_{\lambda 2} - \Delta'_2 \eta'_{\lambda 1}$ is stable for all $\lambda \in \Lambda - \{1, 2\}$.

Let us now write

$$\eta'_{\lambda 1}(s) = a_0(\lambda) + a_1(\lambda)s + \cdots + a_n(\lambda)s^n,$$

$$\eta'_{\lambda 2}(s) = b_0(\lambda) + b_1(\lambda)s + \cdots + b_n(\lambda)s^n.$$

Since the coefficients of $\eta'_{\lambda 1}$ and $\eta'_{\lambda 2}$ are linear combinations of coefficients of $\eta_{\lambda 1}$ and $\eta_{\lambda 2}$, respectively, it follows that $a_i(\lambda)$ and $b_i(\lambda)$ are continuous functions of λ. Since Λ is compact, these functions attain a maximum and a minimum. Assume $a_i(\lambda)\varepsilon[\alpha_i, \beta_i]$, $b_i(\lambda)\varepsilon[\gamma_i, \delta_i]$ for $\alpha_i \leqq \beta_i$, $\gamma_i \leqq \delta_i$, $i = 0, \cdots, n$. Let us denote the columns of the $(2q+2)\times(n+q)$ matrix

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n & & & 0 \\ \gamma_0 & \gamma_1 & \cdots & \gamma_n & & & \\ & \alpha_0 & \cdots & \alpha_{n-1} & \alpha_n & & \\ & \gamma_0 & \cdots & \gamma_{n-1} & \gamma_n & & \\ & & & \vdots & \vdots & & \\ & & & \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ 0 & & & \gamma_0 & \gamma_1 & \cdots & \gamma_n \end{bmatrix} \tag{3.1.6}$$

by $v_0^T, \cdots, v_{n+q-1}^T$, respectively, and the columns of the above matrix (3.1.6) by α_i replaced with β_i, γ_i replaced with δ_i, $i = 0, \cdots, n$, with $u_0^T, \cdots, u_{n+q-1}^T$, respectively. Let us denote Δ'_1/Δ'_2 by

$$\Delta'_1/\Delta'_2(s) = \left[\sum_{i=0}^{q} c_i s^i\right] \Big/ \left[\sum_{i=0}^{q-1} d_i s^i + s^q\right] \tag{3.1.7}$$

where

$$\underline{\psi} \triangleq [c_0 \quad d_0 \quad \cdots \quad c_{q-1} \quad d_{q-1} \quad c_q \quad 1], \tag{3.1.8}$$

$$\underline{s}^T \triangleq [1, s, s^2, \cdots, s^{n+q}]^T. \tag{3.1.9}$$

By Theorem 1.3 of [15] it follows that a sufficient condition for the family of polynomials $\Delta'_1\eta'_{\lambda 2} - \Delta'_2\eta'_{\lambda 1}$ for all $\lambda \in \Lambda - \{1, 2\}$ to be stable is given by the stability of the following four polynomials:

$$\begin{aligned} \alpha_1(s) &= \underline{\psi}[v_0^T v_1^T u_2^T u_3^T v_4^T \cdots]\underline{s}^T, \\ \alpha_2(s) &= \underline{\psi}[u_0^T u_1^T v_2^T v_3^T u_4^T \cdots]\underline{s}^T, \\ \alpha_3(s) &= \underline{\psi}[v_0^T u_1^T u_2^T v_3^T v_4^T \cdots]\underline{s}^T, \\ \alpha_4(s) &= \underline{\psi}[u_0^T v_1^T v_2^T u_3^T u_4^T v_4^T \cdots]\underline{s}^T. \end{aligned} \tag{3.1.10}$$

□

Remark 3.1.5. The four polynomials in (3.1.10) have the same structure as we would obtain as return difference polynomials from four single-input single-output plants compensated by a stable, minimum phase compensator with the exception that the associated $(2q+2)\times(n+q)$ matrices in (3.1.10) are not the Sylvestor matrices corresponding to any single-input single-output plants.

Remark 3.1.6. The proof of the above theorem relies on some recent results due to Kharitonov [28] (see [15] for details).

3.2. Simultaneous stabilization of one-parameter family of plants by interpolation methods. Let $x_1(s)/y_1(s)$ and $x_2(s)/y_2(s)$; $x_1, x_2, y_1, y_2 \in H$ be a pair of proper but not strictly proper plants in $\Omega^n_{1,1}$ written in their coprime representation. For ease of exposition and notational simplicity we assume that the zeros of $x_1y_2 - x_2y_1(s)$ in $\mathbb{C}_u$

are simple and are in the interior of $\mathbb{C}_u$. We now consider a linear one-parameter family of plants described as follows:

(3.2.1) $F = \{g_\lambda(s): g_\lambda(s) = [\lambda x_1 + (1-\lambda)x_2]/[\lambda y_1 + (1-\lambda)y_2], \lambda \in [0, 1], g_\lambda \in \Omega^n_{1,1}\ \forall \lambda\}.$

In order to state the main result of this section we need the following notation.

Let $a_1, \cdots, a_t$ denote the zeros of $x_1y_2 - x_2y_1$ in $\mathbb{C}_u$. Let us define b_i as follows:

(3.2.2) $b_i = x_2/x_1(a_i)$ if the multiplicity of a_i as a common zero of $x_1(s)$ and $x_2(s)$ is less than or equal to the multiplicity of a_i as a common zero of $y_1(s)$ and $y_2(s)$

$= y_2/y_1(a_i)$ otherwise

for $i = 1, \cdots, t$. Let us also define $s_i = (a_i - 1)/(a_i + 1)$ and $z_i = (\sqrt{b_i} - 1)/\sqrt{b_i} + 1)$, where the branch cut for the square root is taken to be the nonpositive real axis. Furthermore let k be the largest real number such that

(3.2.3) $$[1 - k^2 z_i z_j]/[1 - s_i s_j]^t_{i,j=1}$$

is nonnegative definite. The main result of this section is now described.

THEOREM 3.2.1. *The following three statements are equivalent*:

(1) *F is simultaneously stabilizable by some dynamic compensator.*

(2) *F is simultaneously stabilizable by some dynamic compensator of degree $\leqq 3n - 2$.*

(3.2.4) (3) $k > 1$.

Remark 3.2.2. We find Theorem 3.2.1 quite surprising. In fact, whereas the conjecture that "pairs of simultaneously stabilizable plants of bounded McMillan degree have simultaneously stabilizing compensators of bounded McMillan degree" is false (see [13]), the conjecture that "simultaneously stabilizable linear one-parameter families of plants of bounded McMillan degree have simultaneously stabilizing compensators of bounded McMillan degree" is true. In this sense, stabilizing a discrete r-tuple of plants (in particular a pair of plants) simultaneously appears to be a much more difficult problem.

Remark 3.2.3. Results similar to Theorem 3.2.1 are unknown for the multiparameter family of plants.

The following corollary of Theorem 3.2.1 is also quite interesting.

COROLLARY 3.2.4 (Gain margin in problem). The family of plants

(3.2.5) $$\{\lambda g(s): \deg g(s) = n, \lambda \in [0, 1]\}$$

is simultaneously stabilizable if and only if it is simultaneously stabilizable by a compensator of degree $\leqq 3n - 2$.

Thus Theorem 3.2.1 and Corollary 3.2.4 recover a result previously due to Khargonekar and Tannenbaum [27] on the gain margin problem. Furthermore we obtain an explicit bound on the degree of the compensator if one exists.

Proof of Theorem 3.2.1. Let us define

(3.2.6) $$x_\lambda(s) = \lambda x_1(s) + (1-\lambda)x_2(s),$$

(3.2.7) $$y_\lambda(s) = \lambda y_1(s) + (1-\lambda)y_2(s).$$

From Proposition 3.1.1 it follows that a necessary and sufficient condition for the simultaneous stabilization of F is the existence of $\Delta_1, \Delta_2, \Delta_\lambda \in J$, $\lambda \in (0, 1)$ such that

(3.2.8) $$\Delta_2/\Delta_1(a_i) = b_i, \qquad i = 1, \cdots, t,$$

(3.2.9) $$\Delta_2/\Delta_1(\infty) \neq x_2/x_1(\infty),$$

(3.2.10) $$\lambda\Delta_1 + (1-\lambda)\Delta_2(s) = \Delta_\lambda(s)$$

for all $\lambda \in (0, 1)$. Condition (3.2.9) is imposed in order to make proper the simultaneously stabilizing compensator. However, since $[0, 1]$ is compact, we may ignore (3.2.9) according to an argument similar to the proof of Theorem 3.1.2.

Thus a necessary and sufficient condition for the simultaneous stabilization of F is the existence of $\Delta_2/\Delta_1(s)$, which interpolates (a_i, b_i), $i = 1, \cdots, t$ and which does not intersect the nonpositive real axis including infinity at any point in $\mathbb{C}_u$.

Recall that $\mathbb{R}^-$ is the nonpositive real axis including infinity. It is trivial to check that $\mathbb{C}_u$ is conformally equivalent to $\mathbb{D}$ via the following transformation:

$$\psi_1: \quad \mathbb{C}_u \to \mathbb{D}, \qquad s \mapsto \frac{s-1}{s+1}.$$

Likewise $\mathbb{C} - \mathbb{R}^-$ is conformally equivalent to $\mathbb{D}^-$, the open interior of the unit disc, via the following transformation:

$$\psi_2: \quad \mathbb{C} - \mathbb{R}^- \to \mathbb{D}^-, \qquad s \mapsto \frac{\sqrt{s}-1}{\sqrt{s}+1} \tag{3.2.11}$$

where the branch-cut for the square root is taken to be the nonpositive real axis.

Thus the above interpolation problem reduces to the following Nevanlinna-Pick interpolation problem [35], [36]: "Does there exist a rational function with real coefficients which defines a map from $\mathbb{D}$ to $\mathbb{D}^-$ in the complex plane, which interpolates the points

$$\left(\frac{a_i - 1}{a_i + 1}, \frac{\sqrt{b_i} - 1}{\sqrt{b_i} + 1}\right) \tag{3.2.12}$$

for $i = 1, \cdots, t$?"

From standard results in the Nevanlinna-Pick interpolation theory [35], [36] it follows that if (3.2.4) is satisfied there exists a rational function of degree $\leqq t-1$ which solves the above interpolation problem. Working backwards, if (3.2.4) is satisfied there exists a rational function $\Delta_2/\Delta_1(s)$ of degree $\leqq 2t-2$ that satisfies (3.2.8) and (3.2.10). Since the plants x_1/y_1 and x_2/y_2 are of degree n, it follows that if $k > 1$, the simultaneously stabilizing compensator is of degree $\leqq 2t-2-n$. Since $t \leqq 2n$ we have $2t-2-n \leqq 3n-2$. Thus we have (3)$\Rightarrow$(2).

On the other hand, if $k \not> 1$ it follows that the Nevanlinna-Pick interpolation problem cannot be solved even by an analytic function and therefore the one-parameter family of plants F is simultaneously unstabilizable. Thus we have (1)$\Rightarrow$(3).

However, this completes the proof since (2)$\Rightarrow$(1) is clear. □

3.3. Robust simultaneous stabilization methods. The basic question we pose in this section is the following: Given a subset S of plants in $\Omega^\alpha_{m,p}$ (where α is a given fixed integer) which is simultaneously stabilizable by a dynamic compensator? To what extent can S be perturbed to S' in $\Omega \triangleq \bigcup_{\alpha=0}^{\infty} \Omega^\alpha_{m,p}$ such that

$$S \subset S' \subset \Omega \tag{3.3.1}$$

and S' is simultaneously stabilizable?

If p_0 is a nominal plant in S, we shall call S a structured perturbation of p_0. This may result from gross variation of one or more parameters of the plant because of component degradation, a change in the working condition, or failure of one or more components. S', on the other hand, is an unstructured perturbation of p_0. This may be introduced at a high frequency of operation as a result of unmodeled dynamics.

In order to describe the main results of this section we need the following assumptions and definitions.

Assume that $p_0(s)$ is a single-input single-output proper plant of degree $\mu_0 \geqq 1$ and let $r(s)$ be a 1×1 stable proper transfer function of degree μ. Furthermore assume the following:

(1) All the unstable poles of $p_0(s)$ are simple and their real parts are positive.

(2) The relative degree of $r(s)$ is either 0 or 1 and $|r(j\omega)|>0$ for all ω. Let us define the following subset S' of Ω:

$$S' = \{ p(s) : |p(j\omega) - p_0(j\omega)| \leqq |r(j\omega)| \ \forall \omega \text{ and } p(s) \text{ has the same number of unstable poles as does } p_0(s)\}. \tag{3.3.2}$$

Also we define the subfamily S of S' as follows:

$$S \triangleq S' \cap \Omega_{1,1}^{\mu_0+\mu+1}. \tag{3.3.3}$$

The main result of this section is described in the following theorem.

THEOREM 3.3.1. *The following statements are equivalent*:

(3.34) (1) *S' is simultaneously stabilizable by a proper dynamic compensator.*

(3.35) (2) *S is simultaneously stabilizable by a proper dynamic compensator.*

(3.36) (3) *S or S' is simultaneously stabilizable by a proper dynamic compensator of degree $\leqq 2\mu_0+\mu-1$.*

Remark 3.3.2. Theorem 3.3.1 is an adaptation of a recent result due to Kimura [29] and is therefore not entirely surprising. We remark however that Theorem 3.3.1 is the only known result on the simultaneous stabilization of a structured multiparameter family of plants other than the sufficiency condition in Theorem 3.1.2.

Proof of Theorem 3.3.1. Condition (3.3.4) of course implies (3.3.5) since S is a subfamily of S'. To see that (3.3.5) implies (3.3.4) we proceed as follows. Let $C(s)$ be a stabilizer of $p_0(s)$. Assume that S' is simultaneously unstabilizable. It follows from Lemm 1 of [29] that there exists a $p(s)$ in S' such that for some $\omega=\omega_0$ we have

$$|[p(j\omega_0)-p_0(j\omega_0)]C(j\omega_0)]| \geqq |1+p_0(j\omega_0)C(j\omega_0)|. \tag{3.3.7}$$

We now claim that there exists $p^*(s)$ in S such that

$$C(j\omega_0)[p_0(j\omega_0)-p^*(j\omega_0)] = [1+p_0(j\omega_0)C(j\omega_0)]. \tag{3.3.8}$$

Such a plant $p^*(s)$ therefore cannot be stabilized by the controller $C(s)$ since the closed-loop system has a pole at $j\omega_0$. S would therefore be simultaneously unstabilizable by $C(s)$. To prove the claim we proceed as follows.

Using Nevanlinna-Pick interpolation methods [35], [36], we construct a proper rational function $\psi(s)$ with real coefficients of degree 1 with the following property:

$$|\psi(j\omega)| = 1 \quad \forall \omega, \tag{3.3.9}$$

$$\arg \psi(j\omega_0) = \arg[1+p_0(j\omega_0)C(j\omega_0)] - \arg[C(j\omega_0)r_1(j\omega)] \tag{3.3.10}$$

where $r_1(s)$ is any stable, proper rational function of degree μ and $|r_1(j\omega)| = |r(j\omega)|$ for all ω. Let us define

$$p_1(s) = p_0(s) - r_1(s)\psi(s). \tag{3.3.11}$$

It follows from (3.3.7), (3.3.9), and (3.3.11) that

$$\begin{aligned} |C(j\omega_0)[p_0(j\omega_0)-p_1(j\omega_0)]| &= |C(j\omega_0)|\,|r_1(j\omega_0)| \\ &= |C(j\omega_0)|\,|r(j\omega_0)| \\ &\geqq |1+p_0(j\omega_0)C(j\omega_0)|. \end{aligned} \tag{3.3.12}$$

Moreover, using (3.3.10) we have

$$\arg [(C(j\omega_0))[p_0(j\omega_0)-p_1(j\omega_0)]]=\arg [1+p_0(j\omega_0)C(j\omega_0)]. \tag{3.3.13}$$

From (3.3.12), (3.3.13) it follows that there exists an $\varepsilon \in [0, 1]$ such that

$$p^*(s)=p_0(s)-\varepsilon r_1(s)\psi(s) \tag{3.3.14}$$

satisfies condition (3.3.8). Finally $p^*(s)$ clearly belongs to S, implying that S is simultaneously unstabilizable.

The condition (3.3.6) of course implies (3.3.4). To see that (3.3.4)⇒(3.3.6), note that if S' is simultaneously stabilizable, it follows from [29] that there exists a strictly bounded real function $u(s)$ which satisfies an interpolation condition. The degree of $u(s)$ depends only on the total number of interpolating points (see [41, pp. 139–143]) and is therefore one less than the number of unstable poles of $p_0(s)$. In turn, the degree of the compensator (if it exists) is $\leqq 2t+\mu+1$, where t is the number of unstable poles of $p_0(s)$. Clearly $t\leqq \mu_0$ and we have the required inequality. □

4. Simultaneous pole placement of multimode linear dynamical systems by dynamic compensation.

4.1. Statement and motivation of the problem. The pole placement problem arises from an interest in knowing if we can arbitrarily tune the frequency response of a dynamical system by a proper choice of a feedback compensator. Of course if the plant is multi-model, we need to consider an r-tuple of $p\times m$ rational transfer function matrices $G_1(s),\cdots,G_r(s)$ of McMillan degrees $n_1,\cdots,n_r$, respectively, modeling all the various modes of the m-input p-output plant. In fact, the r modes of a strictly proper plant can be topologized as points in the set

$$\Sigma=\Sigma^{n_1}_{m,p}\times\cdots\times\Sigma^{n_r}_{m,p} \tag{4.1.1}$$

where

$$\Sigma^{n_i}_{m,p}=\{p\times m \text{ strictly proper } G_i(s), \deg G_i(s)=n_i\}. \tag{4.1.2}$$

(For details on the above topology, see [7], [23], and [24].) A set $s_1,\cdots,s_{n_i+q}$ of unordered self-conjugate set of complex numbers can be topologized as a point in $\mathbb{R}^{n_i+q}$ via the coefficients of the polynomial

$$\prod_{j=1}^{n_i+q}(s-s_j). \tag{4.1.3}$$

We now address the following problem, which constitutes an important result of this section.

Problem 4.1.1 (Generic simultaneous pole placement at a generic set of poles). Is it true that for all r-tuple $G_1(s),\cdots,G_r(s)$ of $p\times m$ real, strictly proper plants of McMillan degree n_i, $i=1,\cdots,r$, respectively, except perhaps those contained in a proper algebraic set, we can arbitrarily assign a set of r self-conjugate sets of n_i+q poles, $i=1,\cdots,r$, with the possible exception of a proper algebraic subset of poles, of the closed-loop systems $G_i(s)[I+K(s)]^{-1}$, $i=1,\cdots,r$, respectively, by a real, proper compensator $K(s)$ of McMillan degree q?

Roughly speaking, Problem 4.1.1 refers to the question of whether or not it is possible to place the closed-loop poles of almost all r-tuples of plants almost everywhere by a dynamic feedback compensator. Clearly, the exact location of the closed-loop poles is unimportant. Otherwise numerical errors in the location of the closed-loop poles would affect the closed-loop response of the system. Consideration of a generic

r-tuple of plants, although a restriction, enables us to compute explicit bounds on the degree of the compensator. We remark, however, that in this section we say nothing about the pole placement of a nongeneric r-tuple of plants. (For an explicit parameterization of the generic set and a discussion on the nongeneric pole placement problem, see [16].)

As noted in [4], a general method of solving pole placement problems is, first, to parameterize, as an open dense subset of a compact space, compensators with certain of their discrete invariants fixed, and second, to parameterize, for fixed $s_0 \in \mathbb{C}$ and fixed plant $G(s)$, the subset of those compensators $K(s)$ which place a pole of $G(s)[I+K(s)G(s)]^{-1}$ at s_0. Now that we have done this, the pole placement problem becomes a problem of intersection theory on this compact space. For m, p, and q fixed the first phase of this program has been completed in § 2. We now analyze the second phase of this program.

4.2. Analysis of the simultaneous pole placement map. As an immediate extension of the pole placement map in (2.2.5), we define the simultaneous pole placement map

$$\chi: \quad \overline{S^q_{p,m}} - B \to \mathbb{RP}^{n_1+q} \times \cdots \times \mathbb{RP}^{n_r+q} \tag{4.2.1}$$

defined by

$$\chi(K(s)) = ([c_{1,0}, \cdots, c_{1,n_1+q}], [c_{2,0}, \cdots, c_{2,n_2+q}], \cdots, [c_{r,0}, \cdots, c_{r,n_r+q}]) \tag{4.2.2}$$

where

$$c_{i,0} + c_{i,1}s + \cdots + c_{i,n_i+q}s^{n_i+q} \tag{4.2.3}$$

is the characteristic polynomial of the closed-loop system

$$G_i(s)[I+K(s)G_i(s)]^{-1} \tag{4.2.4}$$

for $i = 1, \cdots, r$, respectively. The set B, as before, is the base locus and is described as follows:

$$B \triangleq \bigcup_{i=1}^{r} \bigcap_{H_i} \overline{\chi_i^{-1}(H_i)} \tag{4.2.5}$$

where H_i is a hyperplane in $\mathbb{RP}^{n_i+q}$. Moreover if proj_i is the projection map

$$\text{proj}_i: \quad \mathbb{RP}^{n_1+q} \times \cdots \times \mathbb{RP}^{n_r+q} \to \mathbb{RP}^{n_i+q} \tag{4.2.6}$$

for $i = 1, \cdots, r$ we define

$$\chi_i(\cdot) \triangleq \text{proj}_i \, \chi(\cdot). \tag{4.2.7}$$

Finally $\overline{\chi_i^{-1}(H_i)}$ is the closure of $\chi_i^{-1}(H_i)$ inside $\overline{S^q_{p,m}}$. The base locus B can be equivalently described as the closure of $\{K(s) \in S^q_{p,m}: \det[I+K(s)G_i(s)] \equiv 0$ for some $i \in \{1, \cdots, r\}\}$ where the closure is taken inside $\overline{S^q_{p,m}}$. Problem 4.1.1 can be equivalently described as follows: "For almost all r-tuples of plants in Σ (described in 4.1.1), is it true that the simultaneous pole placement map is almost surjective?"

Remark 4.2.1. If χ is continuous, then almost-surjectivity of χ would imply that χ is surjective. However, since the base locus is not necessarily empty, χ is not, in general, continuous at all points.

Consider a set of r self-conjugate tuples of $n_i + q$ distinct poles given by

$$\{s_{i1}, s_{i2}, \cdots, s_{in_i+q}\} \tag{4.2.8}$$

$i = 1, \cdots, r$. Of course we may define the associated closed-loop characteristic polynomials as follows:

$$(4.2.9) \qquad K_i(s) = \prod_{j=1}^{n_i+q} (s - s_{ij}) = \sum_{j=0}^{n_i+q} c_{ij} s^j$$

for $i = 1, 2, \cdots, r$. The point $[c_{i,0}, \cdots, c_{i,n_i+q}]$ in $\mathbb{RP}^{n_i+q}$ can be defined as the intersection of the hyperplanes $H_{ij}, j = 1, n_i + q$, where H_{ij} is the hyperplane in $\mathbb{RP}^{n_i+q}$ orthogonal to

$$(4.2.10) \qquad [1 \quad s_{ij} \quad s_{ij}^2 \cdots s_{ij}^{n_i+q}].$$

A necessary condition for pole placement at the locations described by (4.2.8) is therefore given by the condition that

$$(4.2.11) \qquad \Omega = \bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i+q} \chi_i^{-1}(H_{ij}) \neq \emptyset.$$

Remark 4.2.2. Even when Ω is nonempty there may not exist a simultaneously pole assigning compensator. This is because Ω may be contained inside the base locus B and therefore for every compensator inside Ω, one of the associated r closed-loop characteristic polynomials would vanish identically. Moreover Ω may contain only points in $\overline{S^q_{p,m}} - S^q_{p,m}$ which are the infinite points of the compactification.

We have therefore established the following theorem.

THEOREM 4.2.3. *Given an r-tuple of proper plants, there exists a finite compensator of degree q which places the poles of the ith plant $G_i(s)$ at conjugate set of distinct $s_{ij}, j = 1, \cdots, n_i + q$, for all $i = 1, \cdots, r$ if and only if*

$$(4.2.12) \qquad (\Omega - B) \cap S^q_{p,m}$$

is nonempty. □

Example 4.2.4. For the purpose of this example, let us assume that the base field is complex. Consider the problem of placing all eight poles, in the closed loop, of a 2×2 proper plant of degree 7 by a 2×2 proper compensator of degree 1. From Example 2.1.2 we know that $S^1_{2,2}$ is parameterized in $\text{Grass}_{\mathbb{C}}(2, 8)$ by a set of four hypersurfaces described by (2.1.17), (2.1.18) and (2.1.19). Moreover the algebraic curve (2.1.17) describes a Schubert hypersurface (see [31]). Equation (2.1.18) is not itself a Schubert hypersurface but it is a hypersurface homologous to a Schubert hypersurface. To see this, let ϕ be the functional in $\mathbb{R}^4$ describing the hypersurface (2.1.18). In the dual space of $\mathbb{R}^4$, ϕ is deformable to a functional ϕ_s describing a Schubert hypersurface. This follows by considering the deformation

$$(4.2.13) \qquad t\phi + (1 - t)\phi_s \quad \text{where } 0 \leq t \leq 1$$

which is clearly an element of $(\mathbb{R}^4)^*$ for all $t \in [0, 1]$. Similarly the quadratic curve (2.1.19) is homologous to a pair of Schubert hypersurfaces.

It has been shown by Byrnes [5] that in order to place all eight poles in the closed loop, we must intersect $S^1_{2,2}$ in $\text{Grass}_{\mathbb{C}}(2, 8)$ by eight Schubert hypersurfaces, each corresponding to a given pole. In other words, the problem reduces to computing the intersection of 11 Schubert hypersurfaces together with a union of two Schubert hypersurfaces in the 12-dimensional $\text{Grass}_{\mathbb{C}}(2, 8)$. The problem is to compute the number of points in the intersection.

It follows from Schubert calculus [31] that if the hypersurfaces are in the "general position," then the number of points in the intersection remains invariant with respect to perturbations. The intersection number is computed by considering the homology class σ of the Schubert hypersurfaces as elements of H^*_{22} (Grass $(m, m+p), Z)$ and computing $2\sigma^{12}$ in the above homology. By Schubert calculus [31], σ^{12} is the generic intersection of 12 Schubert hypersurfaces in $\text{Grass}_{\mathbb{C}}(2, 8)$ and is given by $d(2, 6)$, where

$$d(m,p)=\frac{1!\cdots(p-1)!1!\cdots(m-1)!(mp)!}{1!\cdots(m+p-1)!}. \tag{4.2.14}$$

Thus, if we assume general position, there are at most 264 complex compensators which place the poles. Note that not all of the 264 compensators are necessarily finite; some may even be in the base locus.

Remark 4.2.5. In general, ascertaining whether or not certain points in the intersection of several Schubert hypersurfaces of a Grassmannian are indeed in the base locus or are infinite points is difficult. In the remainder of this section we shall study this problem under the special cases $\min(m, p) = 1$, and in § 5 we shall assume $\min(m, p) \geqq 1$, $q = 0$.

4.3. Analysis of simultaneous pole placement map under the special case $\min(m, p) = 1$. Let us now reinterpret Theorem 4.2.3 under the special case when $\min(m, p) = 1$. We shall see that for this special case, simple linear algebraic techniques enable us to ascertain whether or not (4.2.12) is nonempty. Without any loss of generality let us assume that $m = 1$. Otherwise we can transpose the plants and the compensator. Consider an r-tuple of $p \times 1$ plants of McMillan degree n_i, $i = 1, \cdots, r$, given by

$$\left[\sum_{j=0}^{n_i} d^i_{1,j}s^j, \cdots, \sum_{j=0}^{n_i} d^i_{p,j}s^j\right]^T \left[\sum_{j=0}^{n_i} d^i_{p+1,j}s^j\right]^{-1} \tag{4.3.1}$$

for $i = 1, \cdots, r$. Consider the $1 \times p$ compensator given by

$$\left[\left[\sum_{j=0}^{q} a_{1,j}s^j, \cdots, \sum_{j=0}^{q} a_{p,j}s^j\right]^T \left[\sum_{j=0}^{q} a_{p+1,j}s^j\right]^{-1}\right]^T. \tag{4.3.2}$$

We now ask the following simultaneous pole assignability question.

Question 4.3.1. Do there exist suitable choices of the coefficients $a_{k,j}$, $j = 0, \cdots, q$; $k = 1, \cdots, p+1$ such that

$$\sum_{k=1}^{p+1}\left[\sum_{j=0}^{n_i} d^i_{k,j}s^j\right]\left[\sum_{j=0}^{q} a_{k,j}s^j\right] \equiv \sum_{j=0}^{n_i+q} c_{ij}s^j \tag{4.3.3}$$

for all $i = 1, \cdots, r$ and where c_{ij}s are defined in (4.2.9)?

We would now like to write (4.3.3) as an equation that involves only the coefficients. Consider the notation

$$\underline{a} = \underline{a}_i, \cdots, \underline{a}_{p+1} \tag{4.3.4}$$

where

$$\underline{a}_k = a_{k,0}, \cdots, a_{k,q}. \tag{4.3.5}$$

Furthermore let us define for $k = 1, \cdots, p+1$, the following:

$$D^i_k = \underbrace{\begin{bmatrix} d^i_{k,0} & --- & d^i_{k,n_i} & & \\ & \ddots & & \ddots & \\ & & d^i_{k,0} & & d^i_{k,n_i}\end{bmatrix}}_{\leftarrow\ n_i+q+1\ \rightarrow} \updownarrow q+1, \tag{4.3.6}$$

$$\underline{D}^i = [(D^i_1)^T \ --- \ (D^i_{p+1})^T]^T, \tag{4.3.7}$$

and

$$\underline{s_{ij}} = [1 \quad s_{ij} \quad --- \quad s_{ij}^{n_i+q}]. \tag{4.3.8}$$

We may now write (4.3.3) as

$$\underline{a}\underline{D}^i\underline{s}_{ij}^T = \sum_{l=0}^{n_i+q} c_{il}s_{ij}^l = 0 \qquad (4.3.9)$$

for $j = 1, ---, n_i + q$; $i = 1, ---, r$.

It follows from (4.3.9) that a necessary condition for the existence of a compensator which places the poles of each of the plants $G_i(s)$ at distinct conjugate points $s_{ij}, j = 1, ---, n_i + q$ for all $i = 1, ---, r$ is given by

$$\bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i+q} \operatorname{Ker} \underline{D}^i\underline{s}_{ij} \neq \varnothing. \qquad (4.3.10)$$

Condition (4.3.10) is a necessary and sufficient condition for solvability of (4.3.9) for a suitable vector $\underline{a}$. However, the vector $\underline{a}$ may correspond to a compensator (4.3.2) in the base locus B which needs to be avoided.

The base locus is the set of compensators for which $K_i(s) \equiv 0$ (defined in (4.2.9)) for some $i = 1, \cdots, r$ and is given by

$$\bigcup_{j=1}^{r} \operatorname{Ker} \underline{D}^i. \qquad (4.3.11)$$

Analogously to Theorem 4.2.3 it follows that we have Theorem 4.3.2.

THEOREM 4.3.2. *Assume* $\min(m, p) = 1$. *An r-tuple of proper plants* (4.3.1) *of degree* n_i, $i = 1, \cdots, r$ *is simultaneously pole assignable by a proper, or possibly improper, compensator* (4.3.2) *of degree q at conjugate set of* $n_i + q$ *complex points for all* $i = 1, \cdots, r$ *if and only if*

$$\bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i+q} \operatorname{Ker} \underline{D}^i\underline{s}_{ij} - \bigcup_{i=1}^{r} \operatorname{Ker} \underline{D}^i \qquad (4.3.12)$$

is nonempty. □

Remark 4.3.3. The fact that (4.3.12) is nonempty does not imply that it contains a finite compensator. In fact we must ensure separately that $a_{p+1,q} \neq 0$ in order for the associated compensator to be finite. This is done in § 4.4.

Finally we describe the base locus condition purely as a rank computation under the case $\min(m, p) = 1$ and derive a necessary and sufficient condition when the base locus is empty.

THEOREM 4.3.4. *Assume* $\min(m, p) = 1$. *The base locus associated with the pole placement map* (4.2.1) *for a generic r-tuple of plants is empty if and only if*

$$(q+1)\max(m, p) \leqq \min_{i \in \{1, \cdots, r\}} n_i. \qquad (4.3.13)$$

Proof. The matrix $\underline{D}^i$ is of order $(q+1)(m \times p) \times (n_i + q + 1)$ and is of full rank for a generic r-tuple of plants (see [12], [17] for a proof). Clearly $\operatorname{Ker} \underline{D}^i$ is nonempty for some i if

$$(q+1)(m+p) > n_i + q + 1 \qquad (4.3.14)$$

for some $i = 1, \cdots, r$. □

Remark 4.3.5. Since a necessary condition for simultaneous pole placement of a generic r-tuple of $\min(m, p) = 1$ plants of McMillan degrees n_i, $i = 1, \cdots, r$, respectively, is given by

$$qm + qp + mp \geqq \sum n_i + rq \qquad (4.3.15)$$

we can infer from Theorem 4.3.4 that in most cases of interest, i.e., when (4.3.15) is satisfied, the base locus is nonempty.

To sum up, so far in this section we have studied the simultaneous pole placement problem as an intersection problem in an appropriate compactified space of compensators $\overline{S^q_{p,m}}$. The set of compensators Ω which indeed places the poles simultaneously would in general include infinite compensators and compensators in the base locus. In order to obtain a finite compensator we must exclude the infinite compensators and the base locus from the set Ω. Furthermore we have also analyzed the case $\min(m, p) = 1$ and obtained descriptions of the set Ω, and the base locus B, explicitly.

4.4. Simultaneous pole placement of proper plants by dynamic compensation. In this section we propose to address the simultaneous pole placement Problem 4.1.1. In particular we obtain an upper bound for the degree of the simultaneously pole assignable compensator as a function of n_i, m, p and r. The main result of this section is now stated as follows.

THEOREM 4.4.1. *A generic r-tuple of proper plants* $(G_1(s), \cdots, G_r(s)$, $\deg G_i(s) = n_i$, $i = 1, \cdots, r$, *can be pole assigned arbitrarily by a dynamic compensator provided*

$$r \leqq \max(m, p). \tag{4.4.1}$$

Indeed if (4.4.1) *holds then we have the following*:

(a) *A compensator of McMillan degree q would assign an arbitrary r set of* $n_i + q$ *self-conjugate poles,* $i = 1, \cdots, r$ *provided*

$$(q+1)[\max(m, p) + 1 - r] > \sum_{i=1}^{r} n_i. \tag{4.4.2}$$

(b) *A compensator of McMillan degree q would assign an arbitrary r set of* $n_i + q$ *self-conjugate poles,* $i = 1, \cdots, r$ *except perhaps a proper algebraic subset, provided that*

$$(q+1)[\max(m, p) + 1 - r] \geq \sum_{i=1}^{r} n_i + (1 - r). \tag{4.4.3}$$

It may be noted that even when $r = 1$, (4.4.2) and (4.4.3) are different. This distinction is fundamental and is a reflection of the fact that since the pole placement map is not continuous in general, the exact pole placement and the approximate pole placement problems are different. Thus a generic pair of 1×2 plants of degree one can be pole assignable by a gain feedback almost everywhere. However, for specific pole locations, the associated gain may either be infinite or is in the base locus and therefore cannot be assigned. Indeed, for $r = 1$ we have the following corollary.

COROLLARY 4.4.2. *A generic* $p \times m$ *proper plant of degree n is pole assignable if*

$$(q+1)\max(m, p) \geqq n + 1 \tag{4.4.4}$$

and pole assignable almost everywhere if

$$(q+1)\max(m, p) \geqq n. \tag{4.4.5}$$

In order to prove Theorem 4.4.1, we need the following lemma which also indicates that (4.4.2), (4.4.3) represent tight bounds.

LEMMA 4.4.3. *Assume* $\min(m, p) = 1$. *A necessary and sufficient condition for simultaneous pole placement of a generic r-tuple of proper plants is given by* (4.4.2). *A necessary and sufficient condition for simultaneous pole placement of a generic set of poles for a generic r-tuple of proper plants is given by* (4.4.3).

We now proceed to prove Lemma 4.4.3.

Proof of Lemma 4.4.3. We assume $m = 1$ without any loss of generality and consider the r-tuple of plants given by (4.3.1). There exists a compensator of type (4.3.2) which places the poles of the ith plant $G_i(s)$ at $s_{i,1}, \cdots, s_{i,n_i+q}$ provided that, in the notation of § 4.3,

$$\underline{a}[\underline{D}^1 \underline{D}^2 \cdots \underline{D}^r] = [c_{1,0}, \cdots, c_{1,n_1+q}, c_{2,0}, \cdots, c_{2,n_2+q}, \cdots, c_{r,0}, \cdots, c_{r,n_r+q}] \tag{4.4.6}$$

can be solved for a suitable vector of compensator parameters $\underline{a}$. By a simple algebraic manipulation, we might describe the vector $\underline{a}$ which satisfies (4.4.6) as the set

$$S_1 - S_2 \tag{4.4.7}$$

where

$$S_1 = \bigcap_{i=1}^{r} \operatorname{Ker} \tilde{D}^i, \tag{4.4.8}$$

$$S_2 = \bigcup_{i=1}^{r} \operatorname{Ker} D^i, \tag{4.4.9}$$

$$\tilde{D}^i = \operatorname{col}(\tilde{Q}_1^i, \cdots, \tilde{Q}_{p+1}^i), \tag{4.4.10}$$

$$\tilde{Q}_k^i = \operatorname{diag}(c_{i,n_i+q}, \cdots, c_{i,n_i+q}) \begin{bmatrix} d_{k0}^i & \cdots & d_{kn_i}^i & & 0 \\ & d_{k0}^i & \cdots & & d_{kn_i}^i \\ 0 & & d_{k0}^i & \cdots & d_{kn_i-1}^i \end{bmatrix} - [0, 0, \cdots, 0, d_{kn_i}^i]^T [c_{i0}, \cdots, c_{i,n_i+q}]. \tag{4.4.11}$$

Note that the matrix D^i has already been defined in (4.3.7). Furthermore, note that the set S_2 parameterizes the compensators in the base locus and that $S_1 - S_2$ contains finite or infinite compensators which simultaneously place the poles.

Now that we have described the set $S_1 - S_2$, the following two questions seem to be relevant.

Question 4.4.4. Is the set $S_1 - S_2$ nonempty?

Question 4.4.5. Does the set $S_1 - S_2$ contain vector $\underline{a}$ for which $a_{m+p,q} \neq 0$?

Of course Question 4.4.4 would ascertain if there indeed exists a finite or infinite compensator which simultaneously places the poles. Question 4.4.5 would analyze the possibility of such a compensator being finite. The following two lemmas answer the above two questions.

LEMMA 4.4.6. *The set $S_1 - S_2$ is nonempty if and only if*

$$\text{column span of } D^i \not\subset \text{ column span of } [\tilde{D}^1 \cdots \tilde{D}^r] \tag{4.4.12}$$

for all $i = 1, \cdots, r$.

Proof. (Necessity.) Suppose the contrary. Then there exists $i = i_0$ such that

$$\text{column span of } D^{i_0} \subset \text{column span of } [\tilde{D}^1 \cdots \tilde{D}^r]. \tag{4.4.13}$$

Hence,

$$\operatorname{Ker} D^{i_0} \supset \operatorname{Ker} [\tilde{D}^1, \cdots, \tilde{D}^r] = S_1. \tag{4.4.14}$$

Therefore,

$$S_1 \subset S_2 \tag{4.4.15}$$

so that $S_1 - S_2$ is empty.

(Sufficiency.) By assumption,

$$S_1 \not\subset \operatorname{Ker} D^i \tag{4.4.16}$$

for all $i=1,\cdots,r$, so that $S_1 \not\subset S_2$. Hence, $S_1 \cap S_2$ is a proper subspace of S_1. Thus $S_1 - S_2$ is nonempty. □

LEMMA 4.4.7. *The set $S_1 - S_2$ contains a vector $\underset{\sim}{a}$ for which $a_{m+p,q} \neq 0$ if and only if*

$$S_1 - S_2 \neq \varnothing$$

and

$$(0,0,\cdots,0,1)^T \notin \text{column span of } [\tilde{D}^1,\cdots,\tilde{D}^r]. \tag{4.4.17}$$

Proof. (Necessity.) $S_1 - S_2 \neq \varnothing$ is clearly a necessary condition. Assume on the other hand that the vector $(0,\cdots,0,1)^T \in$ column span of $[\tilde{D}^1,\cdots,\tilde{D}^r]$. Every element of $S_1 - S_2$ is orthogonal to column span of $[\tilde{D}^1,\cdots,\tilde{D}^r]$ and hence orthogonal to $(0,\cdots,0,1)^T$. Thus if $\underset{\sim}{a} \in S_1 - S_2$, then $a_{m+p,q} = 0$.

(Sufficiency.) Suppose on the contrary that every vector $\underset{\sim}{a}$ of $S_1 - S_2$ is such that $a_{m+p,q} = 0$. Hence either $(0,\cdots,0,1)^T \perp S_1 - S_2$ or $S_1 - S_2$ is empty. If $S_1 - S_2$ is nonempty then $S_1 \cap S_2$ is a proper subspace of S_1 and therefore $(0,0,\cdots,0,1)^T \perp S_1$ or equivalently $(0,0,\cdots,0,1)^T \in$ column span of $(\tilde{D}^1,\cdots,\tilde{D}^r)$. □

The proof of Lemma 4.4.3 now follows.

LEMMA 4.4.8. *For some $i \in \{1,2,\cdots,r\}$ if there exists $v \in$ column span of D^i, then there exists some choice of $c_{i,j}$ for which $v \in \tilde{D}^i$.*

Proof. Let $f_0^i, ---, f_{n_i+q}^i$ be the columns of D^i. Let the given vector v be described as

$$v = \alpha_0 f_0^i + --- + \alpha_{n_i+q} f_{n_i+q}^i \tag{4.4.18}$$

for some choice of $\alpha_0,\cdots,\alpha_{n_i+q}$. Let $\tilde{f}_0^i,\cdots,\tilde{f}_{n_i+q-1}^i$ be the columns of $\tilde{D}^i$, where

$$\tilde{f}_j^i = c_{i,n_i+q} f_j^i - c_{i,j} f_{n_i+q}^i \tag{4.4.19}$$

for $j=0,---,n_i+q-1$. Writing (4.4.19) as

$$f_j^i = -\frac{1}{c_{i,n_i+q}} \tilde{f}_j^i + \frac{c_{i,j}}{c_{i,n_i+q}} f_{n_i+q}^i \tag{4.4.20}$$

for $j=0,\cdots,n_i+q-1$ and substituting (4.4.20) in (4.4.18), we have

$$v = \sum_{j=0}^{n_i+q-1} \left[-\frac{\alpha_j \tilde{f}_j^i}{c_{i,n_i+q}} + \frac{\alpha_j c_{i,j}}{c_{i,n_i+q}} f_{n_i+q}^i \right] + \alpha_{n_i+q} f_{n_i+q}^i. \tag{4.4.21}$$

It suffices to show therefore the existence of $c_{i,j}$ such that

$$\sum_{j=0}^{n_i+q} \alpha_j c_{i,j} = 0. \tag{4.4.22}$$

Of course (4.4.22) can be satisfied by some choice of $c_{i,j}$. □

LEMMA 4.4.9. *Assume $\min(m,p)=1$. A generic r-tuple of proper plants can be simultaneously pole assigned by a proper compensator if and only if (4.4.2) is satisfied.*

Proof. (Sufficiency.) For a generic r-tuple of plants $[D^1 \cdots D^r]$ is a matrix of full rank. Thus

$$\text{column span of } D^i \not\subset \text{column span of } [\tilde{D}^1,\cdots,\tilde{D}^r] \tag{4.4.23}$$

for all values of $c_{1,0}, \cdots, c_{1,n_1+q}, \cdots, c_{r,0}, \cdots, c_{r,n_r+q}$ and for all $i=1, \cdots, r$. Hence by Lemma 4.4.6, $S_1 - S_2$ is nonempty for the right-hand side of (4.4.6). Moreover, since (4.4.2) is satisfied, we may conclude for a generic r-tuple of plants that

$$(0, 0, \cdots, 0, 1)^T \notin \text{column span of } (D^1, \cdots, D^r) \tag{4.4.24}$$

so that the hypothesis of Lemma 4.4.7 is also satisfied for the right-hand side of (4.4.6).

(Necessity.) If

$$(q+1)[\max(m, p)+1-r] \leqq \sum_{i=1}^{r} n_i, \tag{4.4.25}$$

then for a generic r-tuple of plants

$$(0, \cdots, 0, 1)^T \in \text{column span of } [D^1, \cdots, D^r]. \tag{4.4.26}$$

Let v_i be the projection of $(0, \cdots, 0, 1)^T$ on the column span of D^i. By Lemma 4.4.8 $v_i \in$ column span of $\tilde{D}^i$ for some choice of $c_{i,j}$, $i=1, \cdots, r$, $j=0, \cdots, n_i+q$. Hence

$$[0, \cdots, 0, 1]^T \in \text{column span of } [\tilde{D}^1, \cdots, \tilde{D}^r] \tag{4.4.27}$$

for some choice of $c_{i,j}$, $i=1, \cdots, r, j=0, \cdots, n_i+q$. It follows from Lemma 4.4.7 that there does not exist a finite proper compensator which places the poles corresponding to the above choice of $c_{i,j}$. □

LEMMA 4.4.10. *Assume* $\min(m, p)=1$. *A generic r-tuple of proper plants can be all simultaneously pole assigned, with the possible exception of a proper algebraic subset of poles, by a proper compensator if and only if* (4.4.3) *is satisfied.*

Proof. (Necessity.) Let us assume that (4.4.3) is not satisfied. For a fixed $i=i_0$, we have

$$\begin{aligned} &\dim [\text{column span of } D^{i0} \cap \text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r)] \\ &\quad \geqq \dim [\text{column span of } D^{i0}] + \dim [\text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r)] \\ &\qquad -(q+1)(\max(m, p)+1). \end{aligned} \tag{4.4.28}$$

It has been shown in [12] that for a generic set of $c_{i,j}-s$,

$$\dim [\text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r)] = \sum n_i + rq.$$

From the assumption we conclude that

$$\dim [\text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r)] - (q+1)(m+p) \geqq 0. \tag{4.4.29}$$

Thus from (4.4.28) and (4.4.29) it may be inferred that generically

$$[\text{column span of } D^{i0}] \subset [\text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r)]. \tag{4.4.30}$$

Hence it follows from Lemma 4.4.6 that for a generic $c_{i,j}$, $S_1 - S_2$ is empty. Hence a pole assigning compensator does not exist.

(Sufficiency.) It is necessary to show that $S_1 - S_2$ is nonempty for a generic $c_{i,j}$. Suppose on the contrary that

$$\text{column span of } D^{i0} \subset \text{column span of } [\tilde{D}^1, \cdots, \tilde{D}^r] \tag{4.4.31}$$

for some $i=i_0$. For a generic $c_{i,j}, j=0, \cdots, n_i+q$

$$\text{column span of } D^{i0} = \text{column span of } [\tilde{D}^{i0}, v_0] \tag{4.4.32}$$

where

$$v_0 = (0, ---, 0, d^{i0}_{1,n_{i0}}, ---, 0, ---, 0, d^{i0}_{r,n_{i0}})^T \tag{4.4.33}$$

or equivalently the matrix

$$[v_0, \tilde{D}^1, \cdots, \tilde{D}^r] \tag{4.4.34}$$

is not of full rank. However, for a generic $c_{i,j}$, (4.4.34) is of full rank which is a contradiction. Thus $S_1 - S_2$ is nonempty for a generic $c_{i,j}$. Moreover, for an algebraic set of $c_{i,j}$

$$[0, \cdots, 0, 1]^T \in \text{column span of } (\tilde{D}^1, \cdots, \tilde{D}^r). \tag{4.4.35}$$

This algebraic set misses the point $c_{i,n_i+q} = 1$, $c_{i,j} = 0$ for all $j = 0, \cdots, n_i$, $i = 1, \cdots, r$, so that it is a proper algebraic set. Thus for a generic set of $c_{i,j}$

$$(0, 0, ---, 0, 1)^T \notin \text{column span of } [\tilde{D}^1, \cdots, \tilde{D}^r]. \tag{4.4.36}$$

Thus by Lemma 4.4.7, $S_1 - S_2$ does indeed contain, a vector $\underline{a}$ for which $a_{m+p,q} \neq 0$. Thus there exists a finite (proper) compensator in $S_1 - S_2$. □

The proof of Lemma 4.4.3 clearly follows from Lemmas 4.4.9 and 4.4.10.

Proof of Theorem 4.4.1. In order to prove Theorem 4.4.1 we need the results of the following two lemmas which are now stated without proof.

LEMMA 4.4.11 [40]. *Given an r-tuple of $p \times m$ plants $G_i(s)$ of degree n_i, each with n_i simple poles, there is an open dense set of $1 \times p$ vectors $v \in \mathbb{R}^p$ such that $vG_i(s)$ has degree n_i for all i.*

LEMMA 4.4.12 [17]. *Given an r-tuple of $p \times m$ plants $G_i(s)$, there exists a constant gain output feedback K such that the closed-loop systems $G_i(s)[I + KG_i(s)]^{-1}$ have distinct simple poles.*

Theorem 4.4.1 now follows by choosing $(v, K) \in \mathbb{R}^p \times \mathbb{R}^{mp}$. We have a mapping from an open dense set

$$\Phi_{(v,K)}: \sum_{m,p}^{n_1} \times \cdots \times \sum_{m,1}^{n_r} \to \sum_{m,1}^{n_1} \times \cdots \times \sum_{m,1}^{n_r}, \tag{4.4.37}$$

$$\Phi_{(v,K)}(G_i(s))_{i=1}^r = (vG_i(s)[1 + KG_i(s)]^{-1})_{i=1}^r \tag{4.4.38}$$

which is rational in the Hankel parameters (H_{ij}) of (G_i). Applying Lemmas 4.4.11, 4.4.12, and (4.4.3) to the case $\min(m, p) = 1$, i.e.,

$$\sum_{m,1}^{n_1} \times --- \times \sum_{m,1}^{n_r} \tag{4.4.39}$$

gives, via composition with Φ, an open dense set of

$$\sum_{m,p}^{n_1} \times --- \times \sum_{m,p}^{n_r} \tag{4.4.40}$$

which can be simultaneously pole assigned. □

4.5. Examples. In this section we consider three illustrative examples.

Example 4.5.1. Let $r = n = m = p = 1$ and $q = 0$, so that (4.4.5) is satisfied and by Corollary 4.4.2 almost all poles can be placed. Let the plant be written as

$$\frac{d_{10} + d_{11}s}{d_{20} + d_{21}s} \tag{4.5.1}$$

and the compensator be written as a_{10}/a_{20}. The sets S_1 and S_2 defined by (4.4.8) and (4.4.9) may be described as

$$S_1 = \text{Ker}\begin{bmatrix} c_{11}d_{10} - c_{10}d_{11} \\ c_{11}d_{20} - c_{10}d_{21} \end{bmatrix}, \tag{4.5.2}$$

(4.5.3) $$S_2 = \text{Ker}\begin{bmatrix} d_{10} & d_{11} \\ d_{20} & d_{21} \end{bmatrix}$$

where

(4.5.4) $$c_{10} + c_{11}s$$

is the associated return difference equation. For a generic plant, S_2 contains the trivial vector so that

(4.5.5) $$S_1 - S_2$$

is nonempty and the base locus is empty. Moreover when

(4.5.6) $$c_{11}d_{10} \neq c_{10}d_{11}$$

the condition of Lemma 4.4.7 is satisfied and there exists a finite compensator which places the poles. Condition (4.5.6) implies that the pole $-d_{10}/d_{11}$ is not assignable and hence, almost all but not all, poles can be placed.

Example 4.5.2. Let us consider the case $r = n = m = p = q = 1$, so that from Corollary 4.4.2 it follows that a generic plant is pole assignable. Let the plant be written as

(4.5.7) $$(d_{10} + d_{11}s)/(d_{20} + d_{21}s)$$

and the compensator be written as

(4.5.8) $$(a_{10} + a_{11}s)/(a_{20} + a_{21}s).$$

The sets S_1, S_2 are given by

(4.5.9) $$S_1 = \text{Ker}\begin{bmatrix} c_{12}d_{10} & c_{12}d_{11} \\ c_{12}d_{20} & c_{12}d_{21} \\ -c_{10}d_{11} & c_{12}d_{10} - c_{11}d_{11} \\ -c_{10}d_{21} & c_{12}d_{20} - c_{11}d_{21} \end{bmatrix},$$

(4.5.10) $$S_2 = \text{Ker}\begin{bmatrix} d_{10} & d_{11} & 0 \\ d_{20} & d_{21} & 0 \\ 0 & d_{10} & d_{11} \\ 0 & d_{20} & d_{21} \end{bmatrix}$$

where the return difference equation is given by

(4.5.11) $$c_{10} + c_{11}s + c_{12}s^2 = 0.$$

For a generic plant, S_2 is nonempty and hence the base locus is nonempty and is of dimension 1. Since S_1 is a subspace of dimension 2 we infer that $S_1 - S_2$ is nonempty. Moreover, for a generic plant the matrix

(4.5.12) $$\begin{bmatrix} 0 & d_{10} & d_{11} & 0 \\ 0 & d_{20} & d_{21} & 0 \\ 0 & 0 & d_{10} & d_{11} \\ 1 & 0 & d_{20} & d_{21} \end{bmatrix}$$

is nonsingular and therefore the condition (4.4.17) is satisfied. Hence there exists a finite compensator which places the poles.

Example 4.5.3. Consider the case $r=m=2$, $q=0$, $p=n_1=n_2=1$. Since the condition (4.4.3) is satisfied, it follows from Theorem 4.4.1 that almost all poles can be simultaneously assignable for a generic pair of plants. Let the plants be

$$(4.5.13)\qquad [d^i_{30}+d^i_{31}s]^{-1}[d^i_{10}+d^i_{11}s \quad d^i_{20}+d^i_{21}s]$$

for $i=1,2$ and let the compensator be

$$(4.5.14)\qquad [a_{10}/a_{30} \quad a_{20}/a_{30}].$$

The sets S_1 and S_2 are given by

$$(4.5.15)\qquad S_1=\operatorname{Ker}\begin{bmatrix} c_{11}d^1_{10}-c_{10}d^1_{11} & c_{21}d^2_{10}-c_{20}d^2_{11} \\ c_{11}d^1_{20}-c_{10}d^1_{21} & c_{21}d^2_{20}-c_{20}d^2_{31} \end{bmatrix}$$

and

$$(4.5.16)\qquad S_2=\operatorname{Ker}\begin{bmatrix} d^1_{10} & d^1_{11} \\ d^1_{20} & d^1_{21} \\ d^1_{30} & d^1_{31} \end{bmatrix}\cup\operatorname{Ker}\begin{bmatrix} d^2_{10} & d^2_{11} \\ d^2_{20} & d^2_{21} \\ d^2_{30} & d^2_{31} \end{bmatrix}.$$

The base locus is a union of two subspaces of dimension 1 for a generic pair of plants. It may also be checked that generically condition (4.4.17) is satisfied so that almost all poles can indeed be placed.

4.6. The stable stabilization problem. It has been shown that to simultaneously stabilize a pair of plants is equivalent to solving a well-known problem considered by Youla, Bongiorno, and Lu [43]: When can a single plant be stabilized by a stable compensator? Motivated by the solution of Youla, Bongiorno, and Lu [43], we consider the following question.

Question 4.6.1. When is a stabilizing compensator of a multi-input multi-output plant stable?

As pointed out by Vidyasagar and Viswanadham [42], the problem of simultaneously stabilizing $r+1$ plants by a stable or unstable compensator is equivalent to the problem of simultaneously stabilizing r plants using a stable compensator (see also Saeks and Murray [37] for $r=1$). It may be pointed out that Question 4.6.1 is interesting in its own right since the unstable compensator might result in poor overall sensitivity of the feedback system to variations in plant parameters.

The following theorem is now stated and proved as an immediate consequence of Theorem 4.4.1 (see [12] for details).

THEOREM 4.6.2. *A generic r-tuple of proper plants* $(G_1(s),\cdots,G_r(s))$, $\deg G_i(s)=n_i$, $i=1,\cdots,r$ *can be stabilized by a stable proper dynamic compensator if*

$$(4.6.1)\qquad r<\max(m,p).$$

Indeed if (4.6.1) *holds then it is enough to choose a compensator of McMillan degree* $=q$, *where q satisfies*

$$(4.6.2)\qquad q[\max(m,p)-r]+\max(m,p)\geqq\sum_{i=1}^{r} n_i.$$

It may be noted that the bound (4.6.1) is consistent with the following lemma due to Vidyasagar and Viswanadham [42].

LEMMA 4.6.3 (Vidyasagar and Viswanadham [42]). *A generic r-tuple of plants is stabilizable generically if and only if a generic* $(r-1)$*-tuple of plants is stabilized by a stable compensator.*

Due to the above lemma the following corollary can be stated.

COROLLARY 4.6.4. *Assume* $\min(m, p) = 1$. *A necessary and sufficient condition for generic stabilization by a stable compensator of an r-tuple of proper or strictly proper plants is given by* (4.6.1).

Proof of Theorem 4.6.2. Consider the compensator in (4.3.2) with the choice of

$$(4.6.3) \qquad (a_{m+p,0}, \cdots, a_{m+p,q})$$

in such a way that

$$(4.6.4) \qquad \sum_{j=0}^{q} a_{m+p,j} s^j$$

is a stable polynomial. By fixing the vector $(a_{m+p,0}, \cdots, a_{m+p,q})$ we can obtain the associated pole placement map χ

$$(4.6.5) \qquad \chi: \quad \mathbb{R}^{(q+1)m} \to \mathbb{R}^{\Sigma n_i + r(q+1)},$$

$$(4.6.6) \qquad \chi(a_{1,0}, \cdots, a_{1,q}, \cdots, a_{m,0}, \cdots, a_{m,q}) = (c'_{1,0}, \cdots, c'_{r,n_r+q})$$

with an associated Sylvestor matrix of order $(q+1)m$ by $\sum n_i + r(q+1)$. By a treatment analogous to the proof of Theorem 4.4.1 we compute the rank of the affine map χ and the result follows. □

Proof of Corollary 4.6.4. Sufficiency is clear from Theorem 4.6.2. To prove necessity, let $r = \max(m, p)$. If stabilizability by a stable compensator is generically satisfied, by Lemma 4.6.3 there exists a generic $(m+p)$-tuple of simultaneously stabilizable $\min(m, p) = 1$ plants. By Theorem 4.4.1 and Lemma 4.4.3, this is a contradiction. □

5. Simultaneous pole placement of multimode linear dynamical systems by constant gain feedback.

5.1. Statement and motivation of the problem. In § 4 we considered the simultaneous pole placement problem by a dynamic compensator and analyzed the case $\min(m, n) = 1$ in considerable detail. In this section we consider gain feedback, i.e., $q = 0$ without the restriction $\min(m, p) = 1$. Specifically, we consider special cases of the Problem 4.1.1 described as follows.

Simultaneous pole placement problem 5.1.1. Given an r-tuple $G_1(s), \cdots, G_r(s)$ of $p \times m$ real, proper transfer functions, $\deg G_i(s) = n_i$, $i = 1, \cdots, r$, does there exist an $m \times p$ real gain K such that the closed-loop systems $G_1(s)[I + KG_1(s)]^{-1}, \cdots, G_r(s) \times [I + KG_r(s)]^{-1}$ have poles in a prescribed r set of the conjugate set of complex numbers $s_{i,1}, s_{i,2}, \cdots, s_{i,n_i+q}$, for $i = 1, \cdots, r$?

A parameterization to the solution of Problem 5.1.1 above is obtained in [14]. In particular, the set of simultaneously pole assignable plants have been described qualitatively as a semialgebraic subset in the space of r-tuples of plants. However, in order to get an indication about the relative size of the space of simultaneously stabilizable r-tuples of plants as a subset of the space of r-tuples of plants we consider the following.

Generic simultaneous pole placement at a generic set of poles 5.1.2. Is it true that for all r-tuples $G_1(s), \cdots, G_r(s)$ of $p \times m$ real, proper transfer functions, except perhaps those contained in a proper algebraic set, we can arbitrarily assign all but possibly a proper algebraic subset of poles of the closed-loop systems $G_i(s)[I + KG_i(s)]^{-1}$, $i = 1, 2, \cdots, r$, by a real gain feedback K?

First, it is quite evident that the conditions of pole placement involve solving a set of simultaneous polynomial equations. It is also clear that the existence of a solution to these equations is equivalent to the surjectivity of the associated pole placement map χ. In this chapter, the pole placement map is analyzed for the various special cases.

Consider an infinitesimal analysis of χ, viz., a calculation of the Jacobian $d\chi$ on k^{mp}, the mp-dimensional space of feedback gains and on a certain submanifold of k^{mp}. By adapting a technique due to Brockett and Byrnes [3] we can show the following.

THEOREM 5.1.3. *A generic r-tuple of strictly proper $p \times m$ real transfer functions of McMillan degree $= n_i$, $i = 1, \cdots, r$, may be pole assigned arbitrarily if*

$$m + p - 1 \geqq \sum_{i=1}^{r} n_i. \tag{5.1.1}$$

Moreover, a generic r-tuple of proper $p \times m$ transfer functions of McMillan degree $= n_i$, $i = 1, \cdots, r$, may be pole assigned arbitrarily if

$$m + p - r - 1 \geqq \sum_{i=1}^{r} n_i. \tag{5.1.2}$$

For the case of a single proper plant, Theorem 5.1.3 specializes to the following corollary.

COROLLARY 5.1.4 (Brockett and Byrnes [3]). *A generic strictly proper $p \times m$ real transfer function of McMillan degree $= n$ may be pole assigned arbitrarily if*

$$m + p - 1 \geqq n. \tag{5.1.3}$$

COROLLARY 5.1.5 (Kimura [30]). *A generic strictly proper $p \times m$ real transfer function of McMillan degree $= n$ is pole assignable almost arbitrarily (i.e., a generic n-tuple of poles may be assigned) if*

$$m + p - 1 \geqq n. \tag{5.1.4}$$

Using the infinitesimal computation, in particular by showing that the Jacobian of the pole placement map is nonsingular at a given gain K_0 and by applying the dominant morphism theorem over $\mathbb{C}$, a technique due to Hermann and Martin [26] has been adapted to show the following theorem.

THEOREM 5.1.6. *A generic r-tuple of $p \times m$ strictly proper complex transfer functions may be assigned a generic $\sum_{i=1}^{r} n_i$-tuple of complex poles by a complex gain feedback K if and only if*

$$mp \geqq \sum_{i=1}^{r} n_i. \tag{5.1.5}$$

COROLLARY 5.1.7 (Hermann and Martin [26]). *A generic $p \times m$ strictly proper complex transfer function may be assigned a generic n-tuple of complex poles by a complex gain feedback K if and only if*

$$mp \geqq n. \tag{5.1.6}$$

Of course over $\mathbb{R}$, (5.1.5) and (5.1.6) give rise to necessary conditions for arbitrary pole assignment.

In an earlier paper [3], Brockett and Byrnes have shown that the number of complex gains which places a distinct self-conjugate set of n poles for a nondegenerate strictly proper $p \times m$ transfer function assuming $mp = n$, counted with multiplicity, is given by $d(m, p)$ defined in (4.2.14). In this section we describe nondegenerate and twisted r-tuples of proper plants and generalize the results presented by Brockett and Byrnes [3] as follows. Consider a set of distinct complex numbers s_{ij}, $i = 1, \cdots, r$, $j = 1, \cdots, n_i$, where for any $i = i_0$ the set $\{s_{i_0 j}\}$ is self-conjugate.

THEOREM 5.1.8. *Assuming $mp = \sum_{i=1}^{r} n_i$ and that there exists an r-tuple of proper plants that are nondegenerate and twisted at s_{ij}, $i = 1, \cdots, r, j = 1, \cdots, n_i$ (to be defined in Definitions 5.3.4 and 5.3.5), the set of complex gains that places the poles s_{ij} for a generic r-tuple of real proper plants is finite and the number is given by $d(m, p)$ (defined in (4.2.14)).*

In particular, the number (4.2.14) is odd if and only if (5.1.7) is satisfied, and we have the following.

COROLLARY 5.1.9. *Under the hypothesis of Theorem 5.1.8, it is possible to assign any arbitrary r set of the self-conjugate set of n_i distinct poles, $i = 1, \cdots, r$, for a generic r-tuple of real proper plants by a real feedback gain if and only if*

$$(5.1.7) \qquad \min(m, p) = 1 \quad or \quad \min(m, p) = 2 \quad and \quad \max(m, p) = 2^r - 1.$$

Remark 5.1.10. The intersection number $d(m,p)$ counts the number of self-conjugate complex gains (finite or infinite) which places the poles provided that the associated Schubert hypersurfaces are in general position. In [3] it was shown that for a nondegenerate $p \times m$ strictly proper transfer function if $mp = n$ then the associated n Schubert hypersurfaces were indeed in general position. In this section we show that if $mp = \sum_{i=1}^{r} n_i$ and if the r-tuple of proper plants are nondegenerate and twisted, then the associated mp Schubert hypersurfaces are in general position and intersect in $d(m, p)$ finite gains.

In general it is difficult to ascertain if an r-tuple of proper plants is nondegenerate and twisted. However, we have the following.

LEMMA 5.1.11. *Assume $m = p = 2$, $n = 1$, $r = 4$. There exists an r-tuple of proper plants that are nondegenerate and twisted.*

In the above case $m = p = 2$, $n = 1$, $r = 4$ if the four Schubert hypersurfaces are in general position, there are $d(m, p) = 2$ complex gains which place the notes. Although two is not an odd number, in this case we are able to show the existence of a real gain which places the poles.

THEOREM 5.1.12. *Assume $m = p = 2$, $n = 1$, $r = 4$. There exists a finite gain K such that the generic simultaneous pole placement for an r-tuple of strictly proper plants is possible.*

In fact it is known from Schubert calculus that under the hypothesis of Theorem 5.1.12 there are exactly two conjugate hyperplanes that are either in the base locus or place the poles. The proof of the above theorem follows by showing that the base locus misses one of the two hyperplanes and that both the gains are real.

Consider, on the other hand, the following special pole placement problem with the assumption that

$$(5.1.8) \qquad s_{i,j} = 0, \quad i = 1, \cdots, r, \quad j = 1, \cdots, n_i.$$

The special pole placement is known as the dead beat control problem, which is stated as follows.

Dead beat control problem 5.1.13. Given an r-tuple of proper plants, $G_1(s), \cdots, G_r(s)$ each of a given McMillan degree n_i, $i = 1, \cdots, r$, with m inputs, p outputs, does there exist a gain K such that $\det[I + KG_i(s)]$ have zeros only at $s = 0$ for every $i = 1, \cdots, r$?

Notice that in the above problem the poles to be placed are not distinct. Hence the Schubert calculus as applied to Theorem 5.1.8 would not be directly applicable. A result due to Anderson and Byrnes [2] has been adapted to show the following theorem.

THEOREM 5.1.14. *A necessary condition for placing all the* $\sum_{i=1}^r n_i$ *poles of an r-tuple of proper* $m \times p$ *plants of McMillan degree* n_i, $i = 1, \cdots, r$, *respectively, at the origin by a gain feedback* K *is given by*

$$mp \geqq \sum_{i=1}^{r} n_i. \tag{5.1.9}$$

COROLLARY 5.1.15 (Anderson and Byrnes [2]). *A necessary condition for placing all the n poles of a generic* $m \times p$ *strictly proper plant* $G(s)$ *at the origin by a gain feedback* K *is given by* $mp \geqq n$.

It is unknown whether (5.1.9) is a necessary condition for generic simultaneous stabilization as well. Present knowledge is limited to $\min(m, p) = 1$, in which case (5.1.9) is known to be a necessary and sufficient condition. On the other hand, by posing the generic simultaneous dead beat control problem, not as a Schubert intersection problem, but as an intersection problem in the associated homology ring of real or complex Grassmannians, we show the following theorem.

THEOREM 5.1.16. *A sufficient condition for simultaneous dead beat control by a real feedback gain of a generic r-tuple of m-input p-output real proper plants is given by* $\min(m, p) = 1$ *or* $\min(m, p) = 2$, $\max(m, p) = 2^r - 1$ *under the condition that* $mp = \sum_{i=1}^r n_i$ *and that there exists an r-tuple of nondegenerate and twisted proper plants.*

We remark that although we conjecture the existence of an r-tuple of nondegenerate and twisted proper plant whenever $mp = \sum_{i=1}^r n_i$, we have only shown this result when $\min(m, p) = 1$ or when $m = p = 2$, $r = 4$, $n_i = 1$, $i = 1, \cdots, 4$.

The main results of this section are now summarized. First, by restricting the simultaneous pole placement map χ onto rank 1 gains and via infinitesimal analysis of χ we obtain a sufficient condition for simultaneous pole assignment of an r-tuple of proper/strictly proper plants by a feedback gain. Furthermore, using infinitesimal analysis of χ over the complex base field and using the dominant morphism theorem, we obtain a necessary and sufficient condition for simultaneous pole assignment by a complex feedback gain. Next we pose the simultaneous pole assignment problem as a Schubert intersection problem and show that if $mp = \sum_{i=1}^r n_i$, then for a nondegenerate and twisted r-tuple of plants, there exists $d(m, p)$ complex conjugate gains which place the poles. If $d(m, p)$ is odd there exists a real gain which places the poles. Even when $d(m, p)$ is not odd we show that for a special case there exists a real gain which places the poles. Finally we analyze the dead beat control problem as a particular case of the pole assignment problem.

5.2. Analysis of the simultaneous pole placement map via rank 1 gains. Let k denote the real or complex field. We now consider a specialization of the simultaneous pole placement map (4.2.1) assuming $q = 0$. To say that the gain K places the poles of $G_1(s), \cdots, G_r(s)$ is to say that

$$\begin{aligned} &\chi(K) = (c'_{1,1}, \cdots, c'_{1,n_1}, \cdots, c'_{r,1}, \cdots, c'_{r,n_r}), \\ &c'_{ij} = c_{ij}/c_{i0}, \qquad i = 1, \cdots, r, \quad j = 1, \cdots, n_i \end{aligned} \tag{5.2.1}$$

is surjective. The coefficients $c_{i,j}$ are defined in (4.2.3) and are also the numerator of the rational function

$$\det[I + KG_i(s)] \quad \text{for all } i = 1, \cdots, r. \tag{5.2.2}$$

Proof of Theorem 5.1.6. Let us represent generically the kjth entry of $G_i(s)$ by $g_{k,j}^{(i)}(s)/g^{(i)}(s)$ where $g^{(i)}(s)$, $g_{k,j}^{(i)}(s)$ are polynomials of degree n_i. The derivative of the pole placement map (4.2.1) is given by

(5.2.3)

$$d\chi|_{K_0}(K) = (\text{coefficients of the numerator of trace}\,[(K+K_0)G_i(s)],\ i=1,\cdots,r).$$

Let $G_{ij}, j=0,1,\cdots$, be the Laurent series coefficients corresponding to $G_i(s)$. Regarding G_{ij} as vectors in k^{mp}, and expressing the map $d\chi|_{K_0}$ with respect to the standard basis for the space of $m\times p$ matrices, if and only if $G_{i0},\cdots,G_{i,n-1}$ are linearly independent, there exists a gain K such that $d\chi$ is of full rank at $K_0=0$. If $mp \geqq \sum_{i=1}^r n_i$, it is possible to realize $G_1(s),\cdots,G_r(s)$, deg $G_i(s)=n_i$ so that $G_{i0},\cdots,G_{i,n-1}$ are linearly independent.

By the dominant morphism theorem [38], a necessary and sufficient condition that the image of χ is almost surjective for generic $G_i(s), i=1,\cdots,r$, is given by (5.1.5). □

LEMMA 5.2.1. *For a $p\times m$ r-tuple of real proper plants $(G_1(s),\cdots,G_r(s))$, deg $G_i(s)=n_i$ chosen generically, there corresponds an r-tuple of generic $1\times(m+p-1)$ real proper transfer functions $\tilde g_i(s)$ of McMillan degree n_i, where $\tilde g_i(s)$ is obtained from*

$$(g_1^{(i)},\cdots,g_{m+p}^{(i)}) \triangleq [1,0,\cdots,0\,|\,1,0,\cdots,0]\begin{bmatrix} G_i(s) & 0 \\ 0 & G_i^T(s)\end{bmatrix} \tag{5.2.4}$$

by deleting $g_{m+1}^{(i)}$ for each $i=1,\cdots,r$.

Proof. A generic r-tuple of $p\times m$ proper plants with distinct simple poles may be written as

$$G_i(s) = \sum_{j=1}^{n_i} R_{i,j}/(s+s_{i,j}) + R_i^{(0)} \tag{5.2.5}$$

where $R_{i,1},\cdots,R_{i,n_i}$ is a self-conjugate set of complex rank 1 matrices for any $i=1,\cdots,r$ and $R_i^{(0)}$ is the value of $G_i(s)$ at infinity. From (5.2.4) and (5.2.5) we may write

$$(g_1^{(i)},\cdots,g_{m+p}^{(i)}) = \sum_{j=1}^{n_i}(k_2^{(0)}R_{ij}\quad k_1^{(0)}R_{ij}^T)/(s+s_{ij}) + (k_2^{(0)}R_i^{(0)}\quad k_1^{(0)}R_1^{(0)T}) \tag{5.2.6}$$

where

$$k_1^{(0)T}k_2^{(0)} = \begin{bmatrix} 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & 0 \end{bmatrix}. \tag{5.2.7}$$

For a fixed $i\in\{1,\cdots,r\}$ and for all $j=1,\cdots,n_i$ all the coefficients in $(k_2^{(0)}R_{i,j}\,|\,k_1^{(0)}R_{i,j}^T)$, $(k_2^{(0)}R_i^{(0)}\,|\,k_1^{(0)}R_i^{(0)T})$, except the $(m+1)$th coefficients, can be obtained arbitrarily for rank 1 matrices $R_{i,j}$ chosen generically. Indeed, for an r-tuple of plants $G_1(s),\cdots,G_r(s)$ chosen generically, the matrices $R_{i,j}, i=1,\cdots,r, j=1,\cdots,n_i$, can be chosen generically so that (5.2.6) takes the form

$$(g_1^{(i)},\cdots,g_m^{(i)},g_1^{(i)},g_{m+2}^{(i)},\cdots,g_{m+p}^{(i)}) \tag{5.2.8}$$

where

$$\tilde g_i(s) = (g_1^{(i)},\cdots,g_m^{(i)},g_{m+2}^{(i)},\cdots,g_{m+p}^{(i)}) \tag{5.2.9}$$

and corresponds to a generic $1\times(m+p-1)$ transfer function of McMillan degree n_i. □

Proof of Theorem 5.1.3. Consider the following notation: If $p(s)/q(s)$ is a proper rational function of degree n, let us denote by $\{p(s)/q(s)\}$ the vector of n coefficients of the numerator of $p'(s)/q'(s) = p(s)/q(s)$ where $p'(s)$ is a monic polynomial of degree n.

Consider the restriction of the simultaneous pole placement map χ to the submanifold M of rank 1 matrices. Let us write a rank 1 matrix K in its dyadic representation

(5.2.10) $$K = k_1^T k_2$$

where

k_1 is a $1 \times m$ nonzero vector,

k_2 is a $1 \times p$ nonzero vector.

Consider the r-tuple of proper plants $G_1(s), \cdots, G_r(s)$. The map χ is written as

(5.2.11) $$\chi(K) = (\{1 + k_2 G_i(s) k_1^T\}, i = 1, \cdots, r).$$

Let M be a submanifold of rank 1 $m \times p$ matrices of dimension $m+p-1$. The derivative $d\chi$ acting on $T_k M$ may be obtained as

(5.2.12) $$d\chi|_{K_0}(K) = \left(\left\{(1, 0, \cdots, 0 | 1, 0, \cdots, 0)\begin{bmatrix} G_i(s) & 0 \\ 0 & G_i^T(s)\end{bmatrix}\begin{bmatrix} k_1^T \\ k_2^T \end{bmatrix}\right\}\right), \qquad i = 1, \cdots, r$$

where K_0 is given by (5.2.7). By (5.2.6) and (5.2.12)

(5.2.13) $$d\chi|_{K_0}(K) = [\{(g_1^{(i)}, \cdots, g_{m+p}^{(i)})(k_1 \quad k_2)^T\}, i = 1, \cdots, r].$$

By scaling the first entry of the vector k_2 as 1 and calling it $k_2' = (1, k_2'')$ it is possible to write, using (5.2.8) and (5.2.9),

(5.2.14) $$d\chi|_{K_0}(K) = (\{\tilde{g}_i(s) k^T + g_1^{(i)}(s)\}, i = 1, \cdots, r)$$

where

(5.2.15) $$k = (k_1 \quad k_2'').$$

A generic $g_i(s)$ may be written as

(5.2.16) $$\tilde{g}_i(s) = (g_1^{(i)\prime}, \cdots, g_m^{(i)\prime}, g_{m+2}^{(i)\prime}, \cdots, g_{m+p}^{(i)\prime})/g_0^{(i)}$$

where

(5.2.17) $$g_j^{(i)} = g_j^{(i)\prime}/g_0^{(i)}, \qquad j = 1, \cdots, m, m+2, \cdots, m+p$$

$g_0^{(i)}, g_j^{(i)\prime}$ for all i, j are polynomials of degree n.

Using the notation in (5.2.16) it is possible to write

(5.2.18) $$d\chi|_{K_0}(K) = \text{coefficients of } (g_1^{(i)\prime}, \cdots, g_m^{(i)\prime}, g_{m+2}^{(i)\prime}, \cdots, g_{m+p}^{(i)\prime})k^T + (g_1^{(i)\prime}, i = 1, \cdots, r).$$

Consider now the matrix representation of the linear map $d\chi|_{K_0}$ in the coefficients of $(g_1^{(i)\prime}, \cdots, g_m^{(i)\prime}, g_{m+2}^{(i)\prime}, \cdots, g_{m+p}^{(i)\prime})$. With respect to the standard basis, we obtain a matrix M_1 of order $(m+p-1)$ by $(\sum n_i + r)$, for generic $g_j^{(i)\prime}$, and for all i, j, the matrix M_1 is of full rank. Hence for a generic r-tuple of proper plants it follows using Lemma 5.2.1 that $d\chi|_{K_0}$ is surjective if

(5.2.19) $$m + p - 1 \geqq \sum n_i + r.$$

If, on the other hand, $G_1(s), \cdots, G_r(s)$ are strictly proper, then $g_j^{(i)}$ for all i, j are polynomials of degree $n-1$ so that the corresponding matrix M is of order $(m+p-1)$ by $(\sum n_i)$. Hence for a generic r-tuple of strictly proper plants $d\chi|_{K_0}$ is surjective if

$$(m+p-1) \geqq n_i. \tag{5.2.20}$$

Finally, by the implicit function theorem, if $d\chi|_{K_0}$ has full rank, then image χ contains an open neighborhood around $\chi(K_0)$. We now claim that χ is surjective. To see that it is, consider scaling either k_2 or k_1 in (5.2.10) by a scale factor λ. By choosing a rank 1 matrix K_0 such that $\chi(K_0)$ contains a neighborhood around the origin and multiplying by an arbitrary scale factor λ, an arbitrary r-tuple of a closed-loop characteristic polynomial can be achieved.

5.3. Simultaneous pole placement as a Schubert intersection problem. In this section we generalize the pole placement strategy introduced by Brockett and Byrnes [3] to analyze the simultaneous pole placement problem. Though repetitive, the main construction is described as follows.

Consider the basic multivariable feedback equations

$$G(s)u = y, \qquad u = -Ky \tag{5.3.1}$$

where u is an m vector and y is a p vector. The closed-loop poles are given by the set of all $s^* \in \mathbb{C}$ where $\det[I + G(s^*)K] = 0$, i.e.,

$$\det \begin{bmatrix} G(s^*) & -I_p \\ I_m & K \end{bmatrix} = 0 \tag{5.3.2}$$

or equivalently

$$\dim [\text{row span of } [G(s^*) \quad -I_p] \cap \text{row span of } [I_m \quad K]] > 0. \tag{5.3.3}$$

Row span of $[G(s^*) \; -I_p]$ is a p plane in $m+p$ space and is therefore a point in Grass $(p, m+p)$. Every plant $G(s)$ gives rise to a curve $\underset{\sim}{G}$: $\mathbb{CP}^1 \to$ Grass $(p, m+p)$ defined by $\underset{\sim}{G}(s) =$ row span of $[G(s) \; -I_p]$. We denote the above curve in the Grassmannian by the graph $[G(s)]$. Likewise, row span of $[I_m \; K]$ is a point in Grass $(m, m+p)$. Note that Grass $(m, m+p)$ is in fact the compactification $\overline{S^0_{p,m}}$ as described in § 2. Not every point in Grass $(m, m+p)$ comes from a feedback gain. The point

$$\text{row span of } [K_1 \quad K_2] \in \text{Grass}\,(m, m+p) \tag{5.3.4}$$

corresponds to the gain $K_1^{-1}K_2$ provided $\det K_1 \neq 0$. On the other hand, the set of points

$$\text{row span of } [K_1 \quad K_2], \qquad \det K_1 = 0 \tag{5.3.5}$$

are the infinite gains in Grass $(m, m+p)$.

If W is a point in Grass $(p, m+p)$ we define $\sigma(W)$ as follows:

$$\sigma(W) \triangleq \{W_1 \in \text{Grass}\,(m, m+p) \mid \dim (W_1 \cap W) > 0\}. \tag{5.3.6}$$

$\sigma(W)$ is classically known as the Schubert hypersurface. It follows from (5.3.3) that $\sigma([G(s^*) \quad -I_p])$ parameterizes the set of gains (finite or infinite) which places the poles of $G(s)$ at $s = s^*$. Thus if the objective is to place the poles at distinct points $s_1, \cdots, s_n$, the pole placement problem is to ensure that

$$\bigcap_{i=1}^{n} \sigma([G(s_i) \quad -I_p]) \tag{5.3.7}$$

contains a finite gain. The contribution of Brockett and Byrnes [3] is that (5.3.7) indeed contains a finite gain for a nondegenerate strictly proper plant.

We now consider Problem 5.1.1 under the assumption $mp=\sum_{i=1}^{r} n_i$. The simultaneous pole placement problem is interpreted as a geometric intersection problem and we ask the following question.

Question 5.3.1.

$$\text{When is } \bigcap_{i=1}^{r}\bigcap_{j=1}^{n_i}\sigma[G_i(s_{i,j})|-I_p] \text{ nonempty?} \tag{5.3.8}$$

DEFINITION 5.3.2. A set of mp, m planes in $m+p$ space W_i, $i=1,\cdots,mp$ are in general position if $\dim \bigcap_{i=1}^{mp}\sigma(W_i)=0$.

Consider an r-tuple of proper plants $G_1(s),\cdots,G_r(s)$ of McMillan degree n_i, $i=1,\cdots,r$. Let s_{ij}, $i=1,\cdots,r$, $j=1,\cdots,n_i$, be the poles to be placed. It is clear that the necessary condition for the simultaneous pole placement problem is that (5.3.8) is nonempty. Conversely, however, if (5.3.8) is nonempty it does not imply that there exists a finite gain K which places the poles. First of all the gain placing the poles might be infinite as defined in (5.3.5) or otherwise the gain K might be in the base locus of the pole placement map, i.e.,

$$\det\begin{bmatrix} N_i(s) & -D_i(s) \\ K_1 & K_2 \end{bmatrix}\equiv 0 \tag{5.3.9}$$

for some $i=1,\cdots,r$.

Remark 5.3.3. Consideration of the base locus for the analysis of the pole placement problem is a new ingredient of this section and does not appear in [3]. This is because the plants we consider are proper as opposed to being strictly proper.

The following two definitions generalize the notion of nondegeneracy introduced in [3].

DEFINITION 5.3.4. An r-tuple of m-input p-output proper plants is nondegenerate at s_{ij} if no Schubert hypersurface $\sigma(W)$ in Grass $(m, m+p)$, where

$$W\in \bigcap_{i=1}^{r}\bigcap_{j=1}^{n_i}\sigma[G_i(s_{i,j})|-I_p], \tag{5.3.10}$$

contains at least one of the curves graph $[G_i(s)]$ for all $i=1,\cdots,r$.

DEFINITION 5.3.5. An r-tuple of m-input p-output proper plants is twisted at $s_{i,j}$ if and only if the set

$$\bigcap_{i=1}^{r}\bigcap_{j=1}^{n_i}\sigma[G_i(s_{i,j})|-I_p]\cap\sigma[0 \quad -I_p] \tag{5.3.11}$$

is empty.

Example 5.3.6. $m=p=n=r=1$.

$$G(s)=(as+b)/(cs+d). \tag{5.3.12}$$

Every $G(s)$ satisfying $ad=bc$ is nondegenerate at any s. Every $G(s)$ satisfying $as_0+b=0$ is twisted at $s=s_0$.

Example 5.3.7. $p=1$, $m=n$, $r=1$.

$$G(s)=\left[\sum_{j=0}^{m} p_j^{(m+1)}s^j\right]^{-1}\left[\sum_{j=0}^{m} p_j^{(1)}s^j,\cdots,\sum_{j=0}^{m} p_j^{(m)}s^j\right]. \tag{5.3.13}$$

Every $G(s)$ satisfying

$$\det\begin{bmatrix} p_0^1 & \cdots & p_m^1 \\ p_0^{m+1} & \cdots & p_m^{m+1}\end{bmatrix}\neq 0 \tag{5.3.14}$$

is nondegenerate at any $s_1, \cdots, s_n$. Every $G(s)$ satisfying

$$\det\begin{bmatrix} \alpha_1(s_1) & \alpha_1(s_m) \\ \alpha_m(s_1) & \alpha_m(s_m) \end{bmatrix} \neq 0 \tag{5.3.15}$$

is twisted at $s_1, \cdots, s_m$, where $\alpha_i(s) = (p_0^i + p_1^i s + \cdots + p_m^i s^m)$, $i = 1, \cdots, m$.

Example 5.3.8. $p = 1$, $n = 1$, $m = r$.

$$G_i(s) = (a_{m+1,i}s + b_{m+1,i})^{-1}(a_{1i}s + b_{1i}, \cdots, a_{mi}s + b_{mi}).$$

The nondegenerate r-tuple is given by

$$\det\begin{bmatrix} a_{11}s_1 + b_{11} & \cdots & a_{m+1,1}s_1 + b_{m+1,1} \\ \vdots & & \vdots \\ a_{1,i-1}s_i + b_{1,i-1} & \cdots & a_{m+1,i-1}s_{i-1} + b_{m+1,i-1} \\ a_{1i} & \cdots & a_{m+1,i} \\ b_{1i} & \cdots & b_{m+1,i} \\ a_{1,i+1}s_i + b_{1i+1} & \cdots & a_{m+1,i+1}s_{i+1} + b_{m+1,i+1} \\ \vdots & & \vdots \\ a_{1m}s_m + b_{1m} & \cdots & a_{m+1,m}s_m + b_{m+1,m} \end{bmatrix} \neq 0, \qquad i = 1, \cdots, m. \tag{5.3.16}$$

The twisted r-tuples are given by

$$\det\begin{bmatrix} a_{11}s_1 + b_{11} & \cdots & a_{m1}s_1 + b_{m1} \\ \vdots & & \vdots \\ a_{1m}s_m + b_{1m} & \cdots & a_{mm}s_m + b_{mm} \end{bmatrix} \neq 0. \tag{5.3.17}$$

Example 5.3.9. This example shows that for $r = 4$, $n = 1$, $m = p = 2$ a generic 4-tuple of strictly proper plants is degenerate. Consider the generic plant

$$G_i: \quad C\mathbb{P}^1 \to \text{Grass}(2,4),$$

$$[s \quad 1] \mapsto \text{row span of} \begin{bmatrix} a & b & s+e & 0 \\ c & d & 0 & s+e \end{bmatrix}, \qquad ad = bc.$$

Clearly the Schubert hypersurface given by

$$\sigma\left[\text{row span of}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right]$$

contains the graph of $G_i(s)$ for $i = 1, 2, 3, 4$, since

$$\det\begin{bmatrix} a & b & s+e & 0 \\ c & d & 0 & s+e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0.$$

Note. This example provides a counterexample to the folklore conjecture that a generic r-tuple ($mp = \sum_{i=1}^r n_i$) of strictly proper plants is nondegenerate.

The following two properties are now established.

LEMMA 5.3.10. *An m-input p-output proper plant $G(s)$ of degree $= n$ is nondegenerate at distinct s_j, $j = 1, \cdots, n$, if and only if graph $G(s)$ is not contained in any Schubert hypersurface.*

Proof. (⇒) Suppose not. Then there exists $W \in \text{Grass}(m, m+p)$ such that $\sigma(W)$ contains row span of $[G(s) \quad -I_p]$ for all s so that

$$W \in \bigcap_{j=1}^{n} \sigma[G(s_j) \quad -I_p] \tag{5.3.18}$$

which is a contradiction since $G(s)$ is nondegenerate at $s_j, j = 1, \cdots, n$.

(⇐) Suppose not. Then there exists $W \in \bigcap_{j=1}^{n} \sigma[G(s_j) \quad -I_p]$ such that $\sigma(W)$ contains graph $[G(s)]$, which is a contradiction. □

LEMMA 5.3.11. *An r-tuple of m-input p-output strictly proper plant is nondegenerate at distinct s_{ij} if and only if it is twisted at distinct s_{ij}, $i = 1, \cdots, r, j = 1, \cdots, n_i$.*

Proof. If the r-tuple of plants is degenerate at s_{ij}, there exists W in Grass $(m, m+p)$ satisfying (5.3.10) and $\sigma(W)$ contains graph $[G_i(s)]$ for some $i = 1, \cdots, r$. Since $G_i(s)$ is strictly proper, $[0 \quad -I_p] \in \sigma(W)$ or

$$W \in \sigma[0 \quad -I_p]. \tag{5.3.19}$$

Thus from (5.3.10) and (5.3.11) it may be concluded that the r-tuple of plants is untwisted at s_{ij}.

Conversely, if the r-tuple of plants is untwisted at s_{ij}, $i = 1, \cdots, r, j = 1, \cdots, n_i$, by defining $s_{i,n+1} = \infty$ for all $i = 1, \cdots, r$ there exists $W \in \text{Grass}(m, m+p)$ such that

$$W \in \bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i+1} \sigma[G_i(s_{ij}) \quad -I_p]. \tag{5.3.20}$$

To say that (5.3.20) is true is to say that the parameters of W satisfy a set of r polynomials in s of degree n_i at $n_i + 1$ different complex points $s_{i1}, \cdots, s_{in_i+1}$. Hence the polynomials must be zero polynomials or equivalently $W \in \sigma[G_i(s) \quad -I_p]$ for all $i = 1, \cdots, r$. Thus the r-tuple of plants is degenerate. □

Finally, consider the following lemma, because of which the set of nondegenerate and twisted r-tuples of plants is interesting to study.

LEMMA 5.3.12. *For an r-tuple of nondegenerate and twisted proper plants, the set of gains contained in the set* (5.3.11) *is finite and places the poles of the ith plant at distinct $s_{i,j}, j = 1, \cdots, n_i$.*

Proof. Suppose not. Then the set (5.3.11) contains

$$W = [K_1 \quad K_2]$$

which either satisfies the condition $W \in \sigma[G_i(s) \quad -I_p]$ for all s and for some $i \in \{1, \cdots, r\}$ or satisfies

$$W \in \sigma[0 \quad -I_p].$$

The former case contradicts the fact that the r-tuple is nondegenerate. The latter case contradicts the fact that the r-tuple is twisted. □

Proof of Theorem 5.1.8. Consider the set

$$V = S_{m,p}^{n_1} \times \cdots \times S_{m,p}^{n_r} \times \text{Grass}(m, m+p) \tag{5.3.21}$$

which topologizes the r-tuple of m-input p-output proper plants together with the finite or infinite compensators. For a fixed $s_{i,j}$, $i = 1, \cdots, r, j = 1, \cdots, n_i$, let us define the algebraic subset of V as follows:

(5.3.22)

$$S = \left\{ G_1(s), \cdots, G_r(s), V \,\middle|\, V \in \bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i} \sigma[G_i(s_{i,j}) | -I_p] \right.$$
$$\left. \text{and graph } G_i(s) \in \sigma(V) \text{ for some } i = 1, \cdots, r \text{ or } [0 \quad -I_p] \in \sigma(V) \right\}.$$

Consider the projection of S onto

$$U = S^{n_1}_{m,p} \times \cdots \times S^{n_r}_{m,p}$$

which is an algebraic set S' since $\text{Grass}_{\mathbb{C}}(m, m+p)$ is projective [19].

Thus the set of r-tuples of complex plants that are either degenerate or untwisted forms an algebraic subset S' of U.

Let k be $\mathbb{R}$ or $\mathbb{C}$. It is well known [24] that U over the field k can be viewed as a manifold of dimension $N = \sum_{i=1}^{r} (n_i(m+p)+mp)$, and hence the algebraic set S' is described locally by a set of algebraic equations. By [14, Lemma 2.2] the same set of algebraic equations describes the complex plants over the complex variables and the real plants over the real variables. Let

$$(5.3.23) \qquad \underset{\sim}{p}(\underset{\sim}{z}) = 0$$

describe the set S'. By restricting $\underset{\sim}{z}$ to its real part, $\underset{\sim}{x} = \text{Re}(\underset{\sim}{z})$, the equation

$$(5.3.24) \qquad \underset{\sim}{p}(\underset{\sim}{x}) = 0$$

describes the set of real plants in S' that is either degenerate or untwisted.

By identity theorem, $\underset{\sim}{p}(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x} \in \mathbb{R}^N$ if and only if $\underset{\sim}{p}(\underset{\sim}{x}) \equiv 0$. However, since by hypothesis there exists one real nondegenerate and twisted plant, it is not in the zero set of $\underset{\sim}{p}(\underset{\sim}{x})$. Hence $\underset{\sim}{p}(\underset{\sim}{x}) \not\equiv 0$, and there is a contradiction. Thus the set

$$(5.3.25) \qquad \{\underset{\sim}{x} \mid \underset{\sim}{p}(\underset{\sim}{x}) = 0\}$$

is a proper algebraic subset of the set of real r-tuple of plants. Hence the set of real nondegenerate and twisted plants is a generic subset of the set of real r-tuple of plants.

Thus by Lemma 5.3.12 there exists a generic real r-tuple of plants for which if the set (5.3.8) is nonempty, it contains finite gains which place the poles of $G_i(s)$ at $s_{i,j}$, $i = 1, \cdots, r, j = 1, \cdots, n_i$.

By a special property of Schubert hypersurfaces [38], we claim that the space X of all gains in the set (5.3.8) has dimension $\geqq mp - \sum_{i=1}^{r} n_i$. It follows that $\dim X \geqq 0$ and we conclude that X is nonempty. In fact, $\dim X = 0$. Otherwise by the same property of Schubert hypersurfaces the space $X \cap \sigma[G_i(s^*) \mid -I_p]$ is nonempty for some $s^* \in \mathbb{C}$ different from $s_{i,j}$. Hence there exists $W \in X$ such that $\sigma(W)$ contains graph $G_i(s)$, which is a contradiction.

Thus $\dim X = 0$ and the $\sum_{i=1}^{r} n_i$ Schubert hypersurfaces are indeed in a general position so that by Schubert calculus [38] the number of finite gains which place the poles is given by (4.2.14). □

Proof of Corollary 5.1.9. The claim clearly follows from the fact that the set of gains which places the poles of $G_i(s)$, at $s_{i,j}$, $i = 1, \cdots, r, j = 1, \cdots, n_i$, is a complex conjugate set. Moreover, the number (4.2.14) is odd if and only if (5.1.7) is satisfied. Finally since the r-tuple of plants is real, and for a fixed i the set $s_{i,j}$ is a complex conjugate set, the set of equations that describes the set (5.3.11) has the property that if row span of $[K_1 \quad K_2]$ is in (5.3.11), then the row span of $[\bar{K}_1 \quad \bar{K}_2]$ the conjugate of the row span of $[K_1 \quad K_2]$ is in (5.3.11). □

LEMMA 5.3.13. *If M_1 and M_2 are two matrices such that rank $(M_1 - M_2) = 1$, then there exists a 2×2 proper rational function $G(s)$ of degree 1 such that $G(s_1) = M_1$ and $G(s_2) = M_2$ for a given s_1, s_2.*

Proof. Writing

$$(5.3.26) \qquad G(s) = R/(s+\lambda) + J$$

we obtain the following pair of simultaneous equations:

$$R/(s_1+\lambda)+J=M_1, \qquad R/(s_2+\lambda)+J=M_2. \tag{5.3.27}$$

Here, R is a rank 1 matrix which may be solved simultaneously to obtain

$$R=[(s_1+\lambda)(s_2+\lambda)/(s_2-s_1)](M_1-M_2). \tag{5.3.28}$$

Since rank $(M_1-M_2)=1$, there exists a rank 1 matrix R satisfying (5.3.28). Using (5.3.26) we can now solve for J. □

Proof of Lemma 5.1.11. For $m=p=2$, $n=1$, $r=4$ there exists an r-tuple of proper plants that is nondegenerate and twisted.

Let M_1, M_2, M_3, M_4 be four 2×2 matrices such that

$$\text{rank}\,(M_i-M_j)\neq 1 \tag{5.3.29}$$

and

$$\bigcap_{i=1}^{4} \sigma[M_i \quad -I]\cap\sigma[0 \quad I] \tag{5.3.30}$$

is empty. Clearly the above property is satisfied generically.

Consider the row span of

$$L_i=[M_i \quad I], \qquad i=1,2,3,4 \tag{5.3.31}$$

as points in Grass (2, 4). By Schubert calculus it is known that

$$\bigcap_{i=1}^{4} \sigma(L_i) \tag{5.3.32}$$

contains exactly two points W_1, W_2 in Grass (2, 4), provided L_i is in general position. Moreover, by considering the cell decomposition of a Grassmannian it can be seen that W_i, $i=1,2$ are irreducible (see Milnor and Stasheff [44]). Define

$$H_j=\{\text{row span of}\,[M \quad I]\,|\,(M-M_j) \text{ is of rank } 1, j=1,2,3,4\} \tag{5.3.33}$$

which is an irreducible Schubert hypersurface in Grass (2, 4).

If $H_j\subset\sigma(W_1)$ then since $\sigma(W_1)$ is irreducible $H_j=\sigma(W_1)$. Hence, $L_1, L_2, L_3, L_4\in H_j$ which, however, contradicts (5.3.29). Hence,

$$H_j-H_j\cap\sigma(W_1) \tag{5.3.34}$$

is nonempty. Since H_j is irreducible, $H_j-H_j\cap\sigma(W_1)$ is dense in H_j. Hence

$$\bigcap_{i=1}^{2} [H_j-H_j\cap\sigma(W_i)] \tag{5.3.35}$$

is nonempty and it is possible to choose a point M_j' in it.

Consider $G_j(s)$ passing through M_j and M_j', where $G_j(s_j)=M_j$, $j=1,2,3,4$. By Lemma 5.3.13 such a $G_j(s)$ indeed exists. Moreover, $G_j(s)$, $j=1,2,3,4$, are clearly nondegenerate. Otherwise either of the plants would be contained in W_1 or W_2, which is a contradiction. Finally the 4-tuple is twisted by (5.3.30). □

Proof of Theorem 5.1.12. Define $\sigma_0(\infty)$ to be the set of infinite gains distinct from

$$[U]=\text{row span of}\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{5.3.36}$$

Let $G_1(s)$, $G_2(s)$, $G_3(s)$, $G_4(s)$ be the four plants and let s_1, s_2, s_3, s_4 be the set of real poles to be placed. By Schubert calculus [31], the set

(5.3.37) $$\bigcap_{i=1}^{4} \sigma[G_i(s_i) \quad -I_2]$$

contains two points for a generic 4-tuple of plants. Consider the set

(5.3.38) $$\Sigma^1_{2,2} \times \Sigma^1_{2,2} \times \Sigma^1_{2,2} \times \Sigma^1_{2,2} \times \text{Grass}\,(2,4)$$

together with the subset

(5.3.39) $$X = \{(G_1, G_2, G_3, G_4, V) \mid G_i(s) \subset \sigma(V), \text{for some } i\}.$$

Consider the projection

(5.3.40) $$p_2\colon \quad X \to \text{Grass}\,(2,4).$$

Let V_g be a generic point in image p_2; then $p_2^{-1}(V_g)$ is a generic fiber at V_g. Moreover, it is known [5] that if V_{g_1} and V_{g_2} are two generic points, then

(5.3.41) $$p_2^{-1}(V_{g_1}) \cong p_2^{-1}(V_{g_2}).$$

First

(5.3.42) $$p_2^{-1}(V_g) = \prod_{i=1}^{4} p_{2i}^{-1}(V_g)$$

where

(5.3.43) $$p_{2i}^{-1}(V_g) = \{G_i(s) \mid G_i(s) \subset \sigma(V_g)\}.$$

Choosing V_g^* to be the

(5.3.44) $$\text{row span of} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

we may describe $p_2^{-1}(V_g^*)$ as follows:

(5.3.45) $$p_2^{-1}(V_g^*) = \left\{(G_1, G_2, G_3, G_4)\colon G_i = \begin{bmatrix} g_{11}^{(i)} & 0 \\ g_{21}^{(i)} & g_{22}^{(i)} \end{bmatrix}\right\}.$$

Since $G_i(s)$ is a 2×2 plant of degree 1 it is clear that

(5.3.46) $$p_{2i}^{-1}(V_g^*) \cong \Sigma^1_{2,1} \cup \Sigma^1_{1,2}$$

and hence

(5.3.47) $$\dim\,(p_{2i}^{-1}(V_g^*)) = 3$$

and

(5.3.48) $$\dim\,(p_2^{-1}(V_g^*)) = 12.$$

Since $\sigma_0(\infty)$ is of dimension 3, it follows that

(5.3.49) $$\dim p_2^{-1}[\sigma_0(\infty)] = 15.$$

Define

(5.3.50) $$\tilde{Y} = \{G_1, G_2, G_3, G_4, V \mid V \in \sigma_0(\infty), \text{ and } G_i \in \sigma(V) \text{ for some } i = 1, 2, 3, 4\}$$

and consider the projection

(5.3.51) $$p\colon \quad \tilde{Y} \to \Sigma^1_{2,2} \times \Sigma^1_{2,2} \times \Sigma^1_{2,2} \times \Sigma^1_{2,2}.$$

From (5.3.49) we conclude that dim $\tilde{Y} = 15$. If $Y =$ image p it follows that $\prod_{i=1}^{4} \Sigma_{2,2}^{1} - Y$ is open and dense. Hence for a generic 4-tuple of plants, there does not exist any element in $\sigma_0(\infty)$ which is in the set (5.3.37).

Since $[U]$ as given by (5.3.36) is in (5.3.37), we have two choices. Either $[U]$ is to be counted twice, or there is a finite gain which places the poles. For a generic r-tuple of plants, the former choice is impossible. To show this, it suffices to construct a particular 4-tuple of plants G_1, G_2, G_3, G_4 for which (5.3.37) contains an element other than $[U]$. □

5.4. Simultaneous dead beat control problem. Consider the following interpretation of the dead beat control problem defined in (5.1.14). Let the state space realization of the plants $G_1(s), \cdots, G_r(s)$ be given by

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t), \\ y(t) &= C_i x(t) + D_i u(t). \end{aligned} \tag{5.4.1}$$

The dead beat control problem may be posed as follows.

Problem 5.4.1. Given an r-tuple of matrix triplets A_i, B_i, C_i, $k = 1, \cdots, r$, does there exist a matrix K such that

$$A_i + B_i K C_i \tag{5.4.2}$$

for all $i = 1, \cdots, r$ are nilpotent?

In this section, Theorem 5.1.12 has been proved by obtaining a necessary condition to Problem 5.4.1. This proof has been adapted from the proof by Anderson and Byrnes [2] for $r = 1$.

Proof of Theorem 5.1.12. Let N_i be the space of the nilpotent $n_i \times n_i$ matrix. It is known that N_i is a subvariety in $\mathbb{R}^{n_i}$ having dimension $n_i^2 - n_i$. For a fixed $B_1, C_1, \cdots, B_r, C_r$ consider the mapping

$$\chi: \quad N_1 \times \cdots \times N_r \times \mathbb{R}^{mp} \to \mathbb{R}^{\sum n_i} \tag{5.4.3}$$

defined by

$$\chi(N_1, \cdots, N_r, K) = (N_1 - B_1 K C_1, \cdots, N_r - B_r K C_r). \tag{5.4.4}$$

By assumption, χ is almost surjective so that

$$\sum (n_i^2 - n_i) + mp \geqq \sum n_i^2 \tag{5.4.5}$$

or, equivalently, (5.1.9) is a necessary condition. □

Proof of dead beat control problem 5.1.13. Stated in other words, an r-tuple of plants is dead beat controllable only in the case where

$$\det \begin{bmatrix} G_i(s) & -I_p \\ I_m & K \end{bmatrix} = 0 \tag{5.4.6}$$

only at $s = 0$, for all $i = 1, \cdots, r$.

Let W be the m plane in Grass$(m, m+p)$ which solves Problem 5.4.1. W may be defined by the common zeros of $\phi_1, \cdots, \phi_p$ on $\mathbb{C}^{m+p}$. Let $g_{ji}(s)$ denote the jth row of

$$[G_i(s) \quad -I_p] \tag{5.4.7}$$

so that

$$(5.4.8)\qquad \begin{bmatrix} \phi_1 \wedge \cdots \wedge \phi_p(g_{1i}(s) \wedge \cdots \wedge g_{pi}(s))|_{s=0} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \frac{d^{n'}}{ds^{n_i}} \phi_1 \wedge \cdots \wedge \phi_p(g_{1i}(s) \wedge \cdots \wedge g_{pi}(s))|_{s=0} = 0 \end{bmatrix}$$

for $i = 1, \cdots, r$. If we embed Grass $(m, m+p)$ by the Plucker embedding

$$(5.4.9)\qquad \text{Grass}\,(m, m+p) \subset \text{Proj}\,(\Lambda^m(\mathbb{C}^{m+p})),$$

then each of the above conditions represents a functional

$$(5.4.10)\qquad \phi_{ij}: \quad \Lambda^m(\mathbb{C}^{m+p}) \dashrightarrow \mathbb{C}$$

for $i = 1, \cdots, r, j = 1, \cdots, n_i$, where $\ker \phi_{ij}$ is a hyperplane intersection of the Grass $(m, m+p)$, i.e., $H_{ij} = \ker \phi_{ij} \cap \text{Grass}\,(m, m+p)$. The $H_{i_j} - s$ are not necessarily Schubert hypersurfaces but have the same homology class. That is, if $\ker \phi$ corresponds to a Schubert hypersurface, then

$$(5.4.11)\qquad [H(\phi)] = [H(\phi_{ij})] \in H_{2mp-2}(\text{Grass}\,(m, m+p), \mathbb{Z}).$$

To show (5.4.11), it is enough to check that the space of linear functionals in $\mathbb{C}^{m+p}$ is path connected and that ϕ can be deformed to ϕ_{ij} by the path

$$(5.4.12)\qquad \begin{aligned} &\psi: \quad [0, 1] \to \mathbb{C}^{m+p*}, \\ &\psi(t) = t\phi + (1-t)\phi_{ij}. \end{aligned}$$

Indeed, $t\phi + (1-t)\phi_{ij} \neq 0$ for every t. Otherwise either ϕ, $\phi_{ij} = 0$ or ϕ and ϕ_{ij} are linearly dependent. Finally by Spianer [39] and Greenberg [18], if the hyperplanes $\ker \phi_{ij}$ are in general position for all i, j then the number of points in Grass $(m, m+p)$ that belongs to H_{ij} is finite and is given by (4.2.14).

Finally, it is claimed that for a nondegenerate and twisted r-tuple of plants, H_{ij} are in general position. Otherwise

$$\dim \bigcap_{i,j} H_{ij} > 0.$$

Hence there exists $[K_1 \quad K_2]$ which satisfies (5.4.8) together with another condition

$$(5.4.13)\qquad \phi_1 \wedge \cdots \wedge \phi_p(g_{1i_0}(s_0) \wedge \cdots \wedge g_{pi_0}(s_0)) = 0$$

for a fixed $i = i_0$ and $s = s_0 \neq 0$. However, (5.4.8) and (5.4.13) together imply that

$$(5.4.14)\qquad \det \begin{bmatrix} G_{i_0}(s) & -I_p \\ K_1 & K_2 \end{bmatrix} \equiv 0$$

since a nonidentically vanishing polynomial of degree n_{i_0} cannot have n_{i_0} roots at the origin together with a root at $s = s_0$. The condition (5.4.14) gives a contradiction to the fact that the r-tuple of plants is nondegenerate.

Finally, since the r-tuple of plants is twisted, we have

$$(5.4.15)\qquad \det \begin{bmatrix} 0 & -I_p \\ K_1 & K_2 \end{bmatrix} \neq 0$$

or $\det K_1 \neq 0$. Hence the set of gains $[K_1 \quad K_2]$ are finite gains. The proof of the theorem now follows analogous to Corollary 5.1.9. □

6. Conclusion. The main purpose of this paper has been to introduce robust system design techniques by using algebraic geometric methods. First, we addressed the question of parameterization and compactified the space of dynamical systems of a given McMillan degree. Next, we considered a family of systems and studied how a family of closed-loop systems degenerates as a function of plant and compensator parameters. Finally, we considered two explicit design problems: the simultaneous stabilization problem and the simultaneous pole placement problem. Many other design problems in this context remain open. We hope that the proposed parameterization of the space of linear dynamical systems will find application in parameter identification and indirect adaptive control problems. In fact, with parameterization of the space of pairs of dynamical systems, the adaptive control problem may be viewed as a degenerating plant/compensator family that is stable in the closed loop.

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