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On minimal degree simultaneous pole assignment problems

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Abstract

In this paper, we show that a generic r-tuple of m-input p-output linear systems is simultaneously pole assignable if $r < m + p$ and the McMillan degrees of the systems are not too different. We also obtain upper bounds for the degrees of the compensators which simultaneously assign the characteristic polynomials of the r-tuple of closed loop systems. The upper bounds are obtained for each of the two cases $r \leqslant \max(m, p)$ and $\max(m, p) < r < m + p$.

Keywords: Linear systems; Simultaneous control; Output feedback pole assignment; Dynamic compensators; Stabilization

1. Introduction

The main objective of this paper is to find the smallest possible integer q such that the closed-loop characteristic polynomials of a generic r-tuple of linear systems of degrees $n_1, \ldots, n_r$, respectively, can be arbitrarily assigned by a single dynamic compensator of degree not exceeding q. Such a problem has been studied by many authors [2,5–9,15–17]. Saeks and Murray [15] considered pairs of single input single output systems and showed that a generic pair of single input, single output systems is not simultaneously stabilizable (hence not simultaneously pole assignable) by a

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dynamic compensator. Motivated by this negative result, Vidyasagar and Viswanadham [17] considered the problem of simultaneous stabilization for m-input, p-output systems, and showed that a generic pair of $p \times m$ systems is simultaneously stabilizable if $\max(m, p) > 1$. At that time, it was unclear that whether a generic r-tuple of plants can be simultaneously stabilizable if $r > 2$. In 1983, Ghosh and Byrnes [8] showed that the number of systems, r, could be chosen as large as $\max(m, p)$, and they showed that a generic r-tuple of $p \times m$ systems is simultaneously pole assignable (hence stabilizable) if $r \leqslant \max(m, p)$. Furthermore they also showed that if $\min(m, p) = 1$, then $r \leqslant \max(m, p)$ is a necessary and sufficient condition for generic simultaneous pole assignment and generic simultaneous stabilization. On the other hand, by counting the dimension of the space of compensators, we can easily verify that if $r \geqslant m + p$, a single compensator cannot simultaneously assign the closed-loop poles of a generic r-tuple of systems to the roots of an arbitrary r-tuple of characteristic polynomials. Therefore a problem that remained open since 1983 is that whether or not a generic r-tuple of $p \times m$ plants is simultaneously pole assignable when $\min(m, p) > 1$ and $\max(m, p) < r < m + p$. Recently, we reported in [9] that the answer to such a question is affirmative if at least $\min(m, p)$ of the McMillan degrees of the systems are not ‘too different’, specifically, if

$$\left| \left\lfloor \frac{n_{i_j} + \sum_{l=\min(m,p)+1}^{r} \lfloor n_{i_l} / \min(m, p) \rfloor}{m + p - r} \right\rfloor - \left\lfloor \frac{n_{i_k} + \sum_{l=\min(m,p)+1}^{r} \lfloor n_{i_l} / \min(m, p) \rfloor}{m + p - r} \right\rfloor \right| \leqslant 1$$

for $1 \leqslant j < k \leqslant \min(m, p)$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. However, because of the restriction on the length of the paper, we were unable to provide the proof in [9]. This paper serves as a follow-up and we prove the above-mentioned result by providing an upper bound for the degree of the simultaneous pole-assigning compensators. We also improve Ghosh and Byrnes’ result in this paper by providing a new upper bound for the degree of the pole-assigning compensator for the case $r \leqslant \max(m, p)$.

The paper is organized as follows. In Section 2, we provide some definitions and preliminary results about polynomial matrices. In Section 3, we formulate the simultaneous pole assignment problem under the behavioral framework [19,20], and define a simultaneous pole assignment map. The main results are proved in Section 4.

2. Polynomial matrices

In this section, we provide some preliminary results about polynomial matrices. Our main reference is [4]. Let $M(s)$ be a $p \times (m + p)$ polynomial matrix over $\mathbb{C}$ with $m > 0$. The ith *row degree* of $M(s)$ is defined as the highest polynomial degree among all the entries in the ith row. The *high degree coefficient matrix* of $M(s)$, denoted by M_{h}, is defined to be the matrix consisting of the coefficients of the monomials whose degrees equal the corresponding row degrees. The *McMillan degree* of a full rank, nonsquare polynomial matrix is defined to be the highest degree of its full size minors. A matrix $M(s)$ is called *row proper* if M_{h} has full rank, and it is called *irreducible* if the full size minors of $M(s)$ are relatively prime.

Proposition 2.1 [4]. *Let $M(s)$ be a full rank (over $\mathbb{C}[s]$), $p \times (m+p)$ polynomial matrix. Then there exists a unimodular $p \times p$ polynomial matrix $U(s)$ such that $U(s)M(s)$ is row proper, and there exists a $p \times p$ polynomial matrix $F(s)$ such that $M(s) = F(s)M_1(s)$ and $M_1(s)$ is irreducible.*

A $p \times (m+p)$ polynomial $M(s)$ is called *minimal* if its rows form a minimal basis of the row space; i.e., they form a basis, and the sum of the row degrees is minimal among all the bases of the row space.

Proposition 2.2 [4]. *A $p \times (m+p)$ polynomial matrix $M(s)$ is minimal if, and only if, it is row proper and irreducible.*

The row degrees of a minimal basis of the row space of $M(s)$ are called the *Forney indices* of $M(s)$.

Remark 2.3. The Forney indices we defined in this paper were called *Kronecker indices* by Fornay in [4] and by some other authors at that time. However, it is customary now to call the row degrees of a row proper matrix, which is unimodular row equivalent to $M(s)$, as the Kronecker indices of $M(s)$ (see [12,13,18]). If $M(s)$ is irreducible, then the Forney indices and the Kronecker indices are equal to each other.

Proposition 2.4 [4]. *Let $M(s)$ be a $p \times (m+p)$ polynomial matrix, and let $y(s) = x(s)M(s)$ be a polynomial $(m+p)$-tuple.*
1. *If $M(s)$ is irreducible, then $x(s)$ must be a polynomial p-tuple.*
2. *If $M(s)$ is row proper and $x(s) = (x_1(s), \dots, x_p(s))$ is a polynomial p-tuple, then*

$$\deg y(s) = \max_{1 \leqslant i \leqslant k} \left\{ \deg x_i + \textit{the } i\text{th } \textit{row degree of } M(s) \right\}.$$

Similar terminologies are also defined for $(m+p) \times p$ matrices if we interchange “row” and “column”. Let $M(s)$ be a minimal matrix. A *dual matrix* of $M(s)$, denoted by $M^{\perp}(s)$, is an $(m+p) \times m$ minimal polynomial matrix such that

$$M(s)M^{\perp}(s) = 0.$$

The Forney indices of $M^{\perp}(s)$ are called the *dual Forney indices* of $M(s)$.

Proposition 2.5 [4]. *The sum of the Forney indices equals the sum of the dual Forney indices.*

Note that the set of all $p \times (m+p)$ polynomial matrices of row degrees at most $(\mu_1, \ldots, \mu_p)$ can be considered as $\mathbb{C}^{(\mu_1+\cdots+\mu_p+p)(m+p)}$, and the set of all matrices of row degrees $(\mu_1, \ldots, \mu_p)$ is a Zariski open set of $\mathbb{C}^{(\mu_1+\cdots+\mu_p+p)(m+p)}$.

Proposition 2.6. *Let $\mathscr{P}$ be the set of all $p \times (m+p)$ polynomial matrices of row degrees $(\mu_1, \ldots, \mu_p)$. Set $n = \mu_1 + \cdots + \mu_p$, and let $k = \lfloor n/m \rfloor$ be the largest integer $\leqslant n/m$, and $d = n - km$ be the remainder of n divided by m. There exists a nonempty Zariski open set $\mathscr{S} \subset \mathscr{P}$ of minimal matrices such that*

1. *every matrix $M(s)$ in $\mathscr{S}$ has the dual Forney indices*

$$\nu_g := \{\underbrace{k, \ldots, k}_{m-d}, \underbrace{k+1, \ldots, k+1}_{d}\}, \tag{2.1}$$

and

2. *for all $M(s) \in \mathscr{S}$, the coefficients of the polynomials in $M^{\perp}(s)$ are rational functions, with nonzero denominators, of coefficients of the polynomials in $M(s)$.*

Proof. Consider the equation $M(s)x(s) = 0$, where $x(s) = x_0 + x_1 s + \cdots + x_i s^i$ is $(m+p)$-tuple column vector of column degree i. Let

$$z_i = \begin{bmatrix} x_0 \\ \vdots \\ x_i \end{bmatrix} \tag{2.2}$$

be a vector consisting of the coefficients of $x(s)$. By setting each s-power term of each entry of $M(s)x(s)$ to be 0, we have a system of

$$\sum_{j=1}^{p} (\mu_j + i + 1) = n + p(i+1)$$

homogeneous linear equations of z_i given by

$$A_i z_i = 0.$$

Note that A_i is an $(n + p(i+1)) \times ((i+1)(m+p))$ matrix, and that we have $n + p(i+1) < (i+1)(m+p)$ if, and only if,

$$i \geqslant k.$$

Therefore the matrix $M(s)$ has dual Forney indices ν_g if, and only if, the columns of A_{k-1} are linearly independent. Such a condition certainly defines a Zariski open subset. Furthermore this Zariski open set is nonempty because one can easily construct a controllable and observable system of McMillan degree n with observability indices $(\mu_1, \ldots, \mu_p)$ and controllability indices ν_g defined by (2.1). For such a system, there exist right and left co-prime factorizations $G(s) = D_r^{-1}(s)N_r(s) = N_l(s)D_l(s)$ such that

$$\begin{bmatrix} D_r(s), -N_r(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} N_l(s) \\ D_l(s) \end{bmatrix}$$

are minimal with row and column degrees $(\mu_1, \ldots, \mu_p)$ and ν_g [4], respectively; i.e., $[D_r(s), -N_r(s)]$ has dual Forney indices ν_g.

To show the second statement, let us write $A_k = [A_{k1}, A_{k2}]$, where A_{k1} and A_{k2} are $(n + p(k+1)) \times (n + p(k+1))$ and $(n + p(k+1)) \times (m-d)$, respectively. If A_{k1} is nonsingular, then the column vectors of

$$\begin{bmatrix} -A_{k1}^{-1}A_{k2} \\ I \end{bmatrix}$$

form a basis of the solution space of $A_k z_k = 0$, which in turn give us $m-d$ independent polynomial vectors of degree k of the solution of $M(s)x(s) = 0$. When $d \neq 0$, we need to find the other d solutions. By re-arranging the order of equations, if necessary, we have

$$A_{k+1} = \begin{bmatrix} A_k & E \\ 0 & H \end{bmatrix},$$

where H is the highest degree coefficient matrix M_{h} of $M(s)$. We partition H into $H = [H_1, H_2, H_3]$, where H_1, H_2, and H_3 are $p \times p$, $p \times d$, $p \times (m-d)$ matrices, respectively, and partition $E = [E_1, E_2, E_3]$ correspondingly. We rewrite

$$A_{k+1} = \begin{bmatrix} A_{k1} & A_{k2} & E_1 & E_2 & E_3 \\ 0 & 0 & H_1 & H_2 & H_3 \end{bmatrix},$$

and if H_1 is nonsingular, the independent columns

$$\begin{bmatrix} A_{k1}^{-1}(E_1 H_1^{-1} H_2 - E_2) \\ 0 \\ -H_1^{-1}H_2 \\ I \\ 0 \end{bmatrix}$$

are solutions of $A_{k+1}z_{k+1} = 0$, which in turn gives us additional d independent polynomial vectors of degree $k+1$.

It follows that if A_{k1} and H_1 are nonsingular, the coefficients of the matrix $M^{\perp}(s) = N_0 + N_1 s + \cdots + N_k s^k + N_{k+1}s^{k+1}$ can be written as

$$\begin{bmatrix} N_0 \\ \vdots \\ N_{k+1} \end{bmatrix} = \begin{bmatrix} -A_{k1}^{-1}A_{k2} & A_{k1}^{-1}(E_1 H_1^{-1} H_2 - E_2) \\ I_{m-d} & 0 \\ 0 & -H_1^{-1}H_2 \\ 0 & I_d \\ 0 & 0 \end{bmatrix}.$$

The set defined by $\det H_1 \neq 0$ is clearly nonempty. We claim that the set defined by $\det A_{k1} \neq 0$ is also nonempty. Note that if $M(s)$ has the dual Forney indices ν_g, then $M^\perp(s)$ has only $m-d$ columns of degree k, and therefore A_k has full rank $n+p(k+1)$. Let $N(s)$ be the sub-matrix of $M^\perp(s)$ consisting of the $m-d$ columns of degree k. Then N_h has full rank $m-d$. Let $\tilde{M}^\perp(s)$ be the matrix obtained by interchanging the rows of $M^\perp(s)$ such that the last $m-p$ rows of corresponding $\tilde{N}_h$ are linearly independent, $\tilde{M}(s)$ be the corresponding matrix obtained by interchanging the columns of $M(s)$ correspondingly, and let C be the $(k+1)(m+p)\times(m-d)$ matrix consisting of the coefficients of $\tilde{N}(s)$ as defined in (2.2). Then from the equation $\tilde{A}_k C = 0$ we have

$$\tilde{A}_{k2} = -\tilde{A}_{k1} C_1 C_2^{-1},$$

where C_1 and C_2 are sub-matrices of C consisting of the first $n+(k+1)p = (k+1)(m+p)-(m-d)$ rows, respectively, the last $(m-d)$ rows. The relation indicates that the columns of $\tilde{A}_{k2}$ are in the column space of $\tilde{A}_{k1}$. Therefore

$$\operatorname{rank} \tilde{A}_{k1} = \operatorname{rank} \tilde{A}_k = n+(k+1)p,$$

and $\det \tilde{A}_{k1} \neq 0$. $\square$

The set of all unimodular column equivalence classes of $(m+p)\times m$ irreducible polynomial matrices of McMillan degree n is a quasi-projective variety [11]. In this quasi-projective variety, the equivalence classes with the Forney indices ν_g defined in (2.1) form a nonempty Zariski open set, and the set of all the other equivalence classes has strictly smaller dimension [13]. For this reason we call the Forney indices ν_g defined in (2.1) the *generic dual indices* of $\mathscr{P}$.

Proposition 2.7 [4]. *Let $M(s)$ be a $p\times(m+p)$ minimal polynomial matrix.*

1. *There exists an $m\times(m+p)$ minimal polynomial matrix $N(s)$ such that*

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$$

is unimodular.

2. *For such $M(s)$ and $N(s)$, there exist dual matrices $M^\perp(s)$ and $N^\perp(s)$ such that*

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \begin{bmatrix} N^\perp(s) & M^\perp(s) \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix}.$$

Proposition 2.8. *Let $M(s)$ and $N(s)$ be $p\times(m+p)$ and $m\times(m+p)$ minimal polynomial matrices, respectively. Then there exist nonzero constants, c_1, c_2, and c_3, such that*

$$\begin{aligned} \det \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} &= c_1 \det M(s) N^\perp(s) \\ &= c_2 \det \left[M^\perp(s), N^\perp(s) \right] \\ &= c_3 \det N(s) M^\perp(s). \end{aligned}$$

Proof. Clearly it is sufficient to prove that

$$\det \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = c_1 \det M(s)N^{\perp}(s).$$

Let $[N^{\perp}(s), Q(s)]$ be the unimodular matrix defined in Proposition 2.7 whose sub-matrices satisfy $N(s)N^{\perp}(s) = 0$ and $N(s)Q(s) = I_m$. It follows that

$$\begin{aligned}\det \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} &= c_1 \det \left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \left[N^{\perp}(s), Q(s) \right] \right) \\ &= c_1 \det \begin{bmatrix} M(s)N^{\perp}(s) & M(s)Q(s) \\ 0 & I_m \end{bmatrix} \\ &= c_1 \det M(s)N^{\perp}(s),\end{aligned}$$

where $c_1 = (\det[N^{\perp}(s), Q(s)])^{-1}$. □

3. Simultaneous pole assignment map

Let us consider a linear system

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx + Du,\end{aligned}$$

together with a dynamic compensator

$$\begin{aligned}\dot{z} &= Ez + Fy, \\ u &= Hz + Ky.\end{aligned}$$

The closed loop system is described as

$$\begin{bmatrix} (\mathrm{d}/\mathrm{d}t)I - A & 0 & -B & 0 \\ C & 0 & D & -I \\ 0 & (\mathrm{d}/\mathrm{d}t)I - E & 0 & -F \\ 0 & H & -I & K \end{bmatrix} \begin{bmatrix} x \\ z \\ u \\ y \end{bmatrix} = 0$$

and the closed loop characteristic polynomial is given by

$$\det \begin{bmatrix} sI - A & 0 & -B & 0 \\ C & 0 & D & -I \\ 0 & sI - E & 0 & -F \\ 0 & H & -I & K \end{bmatrix}$$

provided that it is a nonzero polynomial. The map sending each dynamic compensator to the corresponding characteristic polynomial of the closed loop system is called the *pole assignment map*.

There is also a higher-order representation of the pole assignment map. If (A, C) and (E, H) are observable, then the polynomial matrices

$$\begin{bmatrix} sI - A \\ C \end{bmatrix}, \quad \begin{bmatrix} sI - E \\ H \end{bmatrix}$$

are minimal. It follows from Proposition 2.7 that there exists a unimodular matrix

$$\begin{bmatrix} M_{11}(s) & M_{12}(s) & 0 & 0 \\ M_{21}(s) & M_{22}(s) & 0 & 0 \\ 0 & 0 & N_{11}(s) & N_{12}(s) \\ 0 & 0 & N_{21}(s) & N_{22}(s) \end{bmatrix}$$

such that

$$\begin{bmatrix} M_{11}(s) & M_{12}(s) & 0 & 0 \\ M_{21}(s) & M_{22}(s) & 0 & 0 \\ 0 & 0 & N_{11}(s) & N_{12}(s) \\ 0 & 0 & N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} sI - A & 0 & -B & 0 \\ C & 0 & D & -I \\ 0 & sI - E & 0 & -F \\ 0 & H & -I & K \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0 & * & * \\ 0 & 0 & P_1(s) & P_2(s) \\ 0 & I_q & * & * \\ 0 & 0 & R_1(s) & R_2(s) \end{bmatrix}.$$

Therefore the closed loop poles are the zeros of the polynomial

$$\det \begin{bmatrix} P_1(s) & P_2(s) \\ R_1(s) & R_2(s) \end{bmatrix} := \det \begin{bmatrix} P(s) \\ R(s) \end{bmatrix}.$$

Remark 3.1. The rational functions $-P_2^{-1}(s)P_1(s)$ and $-R_1^{-1}R_2(s)$ are left factorizations of the transfer functions of the plant and compensator. They are left co-prime factorizations if, and only if, the first-order representations are also controllable. If we define

$$w = \begin{bmatrix} u \\ y \end{bmatrix},$$

then $P(\mathrm{d}/\mathrm{d}t)w(t) = 0$ and $R(\mathrm{d}/\mathrm{d}t)w(t) = 0$ are known as the *kernel representations* (also autoregressive representations as described in [20]) of the plants and compensators.

Using Proposition 2.8, it follows that if $R(s)$ is irreducible, then the closed loop poles are also the zeros of det $P(s)R^{\perp}(s)$.

Remark 3.2. If we write

$$R^{\perp}(s) = \begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix},$$

where $Q_2(s)$ is $p \times p$, then $Q_1(s)Q_2^{-1}(s)$ is a right co-prime factorization of the transfer function of the compensator. The representation $w(t) = R^{\perp}(\mathrm{d}/\mathrm{d}t)v(t)$ is called the *image representation* (also moving average representation as described in [19]) of the compensators, where $v(t) = M(\mathrm{d}/\mathrm{d}t)w(t)$ has been defined using the polynomial matrix $M(s)$ such that

$$\begin{bmatrix} R(s) \\ M(s) \end{bmatrix}$$

is unimodular. Such a representation is well known under behavioral framework.

Two $p \times (m + p)$ polynomial matrices $P(s)$ and $\hat{P}(s)$ are called *rational unimodular row equivalent* if there exists a $p \times p$ rational matrix $U(s)$, $\det U(s)$ is a nonzero constant, such that

$$P(s) = U(s)\hat{P}(s).$$

Rational unimodular column equivalence is defined the similar way.

Let $K^n_{p,m}$ be the set of all rational unimodular row equivalence classes of $p \times (m + p)$ polynomial matrices of McMillan degree $\leqslant n$, and $\tilde{K}^q_{p,m}$ be the set of all rational unimodular column equivalence classes of $(m + p) \times p$ polynomial matrices of McMillan degree $\leqslant q$. Then $K^n_{p,m}$ and $\tilde{K}^q_{p,m}$ are projective varieties, and they are compactifications of the set of all *m*-input, *p*-output systems of McMillan degrees at most *n* and the set of all *p*-input, *m*-output dynamic compensators of McMillan degrees at most *q* (see [11,14]). Furthermore, if a polynomial matrix is irreducible, then its rational unimodular equivalence class coincides with its polynomial unimodular equivalence class. Therefore, there is a one to one correspondence between the equivalence classes of irreducible polynomial matrices and controllable and observable systems. The set of all equivalence classes of irreducible polynomial matrices is certainly a nonempty Zariski open subset of $K^n_{p,m}$.

Remark 3.3. The projective variety $K^n_{p,m}$ has singularities. There is also a smooth compactification of the set of all *m*-input, *p*-output systems of McMillan degrees at most *n* called the Grothendieck Quot-scheme. In terms of matrices this is achieved through the concept of homogenous autoregrassive systems as introduced in [10].

Let us consider a set of *r* controllable and observable systems described in kernel representations by

$$P_i(\mathrm{d}/\mathrm{d}t)w(t) = 0, \quad i = 1, \ldots, r,$$

for some irreducible polynomial matrices $P_i(s) \in K^{n_i}_{p,m}$ of McMillan degree n_i, and let a compensator be given in image representation by

$$w(t) = Q(\mathrm{d}/\mathrm{d}t)v(t)$$

for an irreducible polynomial matrix $Q(s) \in \tilde{K}^q_{p,m}$. It follows from above that the closed loop systems are given by

$$P_i(\mathrm{d}/\mathrm{d}t)Q(\mathrm{d}/\mathrm{d}t)v(t) = 0, \quad i = 1, \ldots, r,$$

and the closed loop characteristic polynomials are given by

$$\det P_i(s)Q(s), \quad i = 1, \ldots, r,$$

provided that none of them are identically zero polynomials.

Definition 3.4. The simultaneous pole assignment map $\hat{\chi} : \tilde{K}^q_{p,m} \to \mathbb{P}^{n_1+q} \times \cdots \times \mathbb{P}^{n_r+q}$ is defined by

$$\hat{\chi}(Q) = \big(\det P_1(s)Q(s), \ldots, \det P_r(s)Q(s)\big), \tag{3.1}$$

where a polynomial $a_0 + a_1 s + \cdots + a_k s^k$ is identified with a point $(a_0, a_1, \ldots, a_k) \in \mathbb{P}^k$.

Note that $\hat{\chi}$ is a rational map, and it is not defined at the point where $\det P_i(s)Q(s) = 0$ for some i.

Definition 3.5. A compensator $Q(s) \in \tilde{K}^q_{p,m}$ is called a simultaneous dependent compensator if

$$\det P_i(s)Q(s) \equiv 0, \quad i = 1, \ldots, r.$$

Proposition 3.6 [9]. *If $r \leqslant \max(m, p)$, then a simultaneous dependent compensator of degree at most q exists where q is the smallest integer which satisfies*

$$q + \big(\lfloor q/\min(m, p)\rfloor + 1\big)\big(\max(m, p) - r\big) \geqslant \sum_{i=1}^{r} \lfloor n_i/\min(m, p)\rfloor, \tag{3.2}$$

where $\lfloor x\rfloor$ is the largest integer $\leqslant x$.

Remark 3.7. From the proof of Proposition 3.6 in [9] we claim that (assume $p \leqslant m$) the simultaneous dependent compensator $Q(s)$ in Proposition 3.6 consists of p columns of lowest degrees of

$$\begin{bmatrix} \alpha_1(s) \\ \vdots \\ \alpha_r(s) \end{bmatrix}^{\perp},$$

where each $\alpha_i(s)$ corresponds to the lowest degree row in the minimal polynomial matrix $P_i(s)$.

The simultaneous dependent compensator for $m < p$ can be constructed likewise if we use the image representations of the systems (which are $(m + p) \times m$ polynomial matrices) and kernel representations of the compensators (which are $m \times (m + p)$ polynomial matrices).

Proposition 3.8 [9]. *If* $\min(m, p) < r \leqslant m + p - 1$, *then a simultaneous dependent compensator of degree at most*

$$q = \sum_{i=1}^{\min(m,p)} \left\lfloor \frac{n_i + \sum_{j=\min(m,p)+1}^{r} [n_j / \min(m, p)]}{m + p - r} \right\rfloor$$

exists.

Remark 3.9. The simultaneous dependent compensator in Proposition 3.8 is constructed as follows. Without any loss of generality assume that $p \leqslant m$. For $i = p + 1, \ldots, r$, let $\alpha_i(s)$ be the row of $P_i(s)$ with lowest degree, and for $i = 1, \ldots, p$, let β_i be the lowest degree column vector of

$$\begin{bmatrix} P_i(s) \\ \alpha_{p+1}(s) \\ \vdots \\ \alpha_r(s) \end{bmatrix}^{\perp}.$$

Then we conclude that $Q = [\beta_1(s), \ldots, \beta_p(s)]$ is a dependent compensator. We may have to replace any linearly dependent vector by an arbitrary vector in the dual space of span $\{\alpha_{p+1}, \ldots, \alpha_r\}$.

We can also define an affine simultaneous pole assignment map. Let $\mathscr{M}_q$ be the set of all $(m + p) \times p$ polynomial matrices whose sum of row degrees is at most q. Then every equivalence class of $\tilde{K}^q_{p,m}$ has a matrix in $\mathscr{M}_q$.

Definition 3.10. The affine simultaneous pole assignment map

$$\chi : \mathscr{M}_q \to \mathbb{R}^{n_1+\cdots+n_r+rq+r}$$

is defined by

$$\begin{aligned} \chi(Q) &= \big(\chi_1(Q), \ldots, \chi_r(Q)\big) \\ &= \big(\det P_1(s)Q(s), \ldots, \det P_r(s)Q(s)\big), \end{aligned} \tag{3.3}$$

where a polynomial $a_0 + a_1 s + \cdots + a_k s^k$ is identified with a point $(a_0, a_1, \ldots, a_k) \in \mathbb{R}^{k+1}$.

Remark 3.11. Clearly $\hat{\chi}$ is onto (almost onto) if, and only if, χ is onto (almost onto).

The importance of the simultaneous dependent compensator is indicated by the next result. Let $\nu = (\nu_1, \ldots, \nu_p)$ be the column degrees of a matrix in $\mathscr{M}_q$, and let $l = [q/p]$ and $e = q - lp$. Let us also define

$$\mathscr{M}^g_q = \left\{ Q(s) \in \mathscr{M}_q \,\middle|\, \begin{array}{ll} \nu_i \leqslant l & \text{if } 1 \leqslant i \leqslant p - e \\ \nu_i \leqslant l + 1 & \text{if } p - e + 1 \leqslant i \leqslant p \end{array} \right\}. \tag{3.4}$$

Then $\mathscr{M}_q^g$ is an affine space of dimension $(m+p)(p+q)$.

Proposition 3.12. *The simultaneous pole assignment map is onto if there is a simultaneous dependent compensator $Q(s) \in \mathscr{M}_q^g$ such that the Jacobian*

$$d\chi_Q : \mathscr{M}_q^g \to \mathbb{R}^{n_1+\cdots+n_r+rq+r}$$

of χ at $Q(s)$ is onto.

Proof. Under the given condition, χ maps a small neighborhood of $Q(s)$ in $\mathscr{M}_q^g$ onto a small neighborhood of 0 by the inverse function theorem. Since χ is homogeneous, the whole $\mathbb{R}^{n_1+\cdots+n_r+rq+r}$ is contained in $\chi(\mathscr{M}_q^g)$. $\square$

Proposition 3.13. *Foreach $Q(s) \in \mathscr{M}_q^g$, theJacobian $d\chi_Q : \mathscr{M}_q^g \to \mathbb{R}^{n_1+\cdots+n_r+rq+r}$ is given by*

$$d\chi_Q(X(s)) = \big(\mathrm{tr}\,(R_1(s)X(s)), \ldots, \mathrm{tr}\,(R_r(s)X(s))\big),$$

where

$$R_i(s) = \mathrm{adj}\big(P_i(s)Q(s)\big)P_i(s).$$

The proof of this result is similar to the proof of Theorem 3.10 in [14].

4. Generic simultaneous pole assignability

We are now ready to prove the main results of this paper. We prove two lemmas about polynomial matrices first. A subset is called generic if it contains a nonempty Zariski open set. The elements in a generic set are called generic elements.

Lemma 4.1. *Let $P(s)$ be a $p \times (m+p)$ minimal polynomial matrix, and let r be a positive integer less than m, $\nu_1, \ldots, \nu_r$ be any nonnegative integers, and $Q(s)$ be an $r \times (m+p)$ polynomial matrix of row degrees $\nu_1, \ldots, \nu_r$ such that*

$$\begin{bmatrix} Q(s) \\ P(s) \end{bmatrix}$$

is a minimal polynomial matrix. Then the set of all such $Q(s)$'s form a nonempty Zariski open subset of the affine space of all $r \times (m+p)$ polynomial matrices of row degrees $\nu_1, \ldots, \nu_r$.

Proof. Certainly a set of such polynomial matrices $Q(s)$ would form a Zariski open subset. So we only need to show that this set is nonempty. It is sufficient to prove the result for $r = 1$, and also for $m = 2$ because we can always consider the sub-matrix of $P(s)$ consisting of $p+2$ columns. Furthermore, we can prove the result for

$$P(s)T \tag{4.1}$$

for some $T \in \mathrm{GL}(m+p)$ instead of $P(s)$ itself. So without loss of generality, we assume that:

1. The polynomial matrix $P(s) = [N(s), -D(s)]$ defines a kernel representation of an 2-input, *p*-output, controllable, observable, strictly proper linear system of degree *n*, where *n* is the sum of the row degrees of $P(s)$.
2. The system is controllable through the first input channel. One can always achieve this by applying output feedback and changing the basis of input space [3]. This operation is equivalent to transformation of the type (4.1).

Let

$$\begin{aligned} \dot{x} &= Ax + b_1u_1 + b_2u_2, \\ y &= Cx \end{aligned}$$

be a state space representation of the system $P(s)$ with controllable (A, b_1).

It is sufficient to show that for any ν, there exists a dynamic compensator

$$\begin{aligned} \dot{z} &= Fz + Gy, \quad z \in \mathbb{R}^{\nu}, \\ u_2 &= hz \end{aligned}$$

such that the combined system

$$\begin{aligned} \dot{\begin{bmatrix} x \\ z \end{bmatrix}} &= \begin{bmatrix} A & b_2h \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u_1 \\ \begin{bmatrix} y \\ u_2 \end{bmatrix} &= \begin{bmatrix} C & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned} \tag{4.2}$$

is controllable and observable.

The above result is obviously true for $\nu = 0$. So we consider the case when $\nu \geqslant 1$. The combined system is observable when $G = 0$ and when (F, h) is observable. So it is observable for the generic (F, G, h). We now show that when $h = 0$, we can always choose F and G such that

$$(\hat{A}, \hat{b}) := \left(\begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}, \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \right)$$

is controllable. By applying state feedback (which will not change the controllability) we can assume $A^n = 0$. Let $0 \leqslant k \leqslant n-1$ be the largest number such that $CA^k b_1 \neq 0$. It follows that

$$\hat{A}^i \hat{b} = \begin{bmatrix} A^i b_1 \\ * \end{bmatrix}, \quad i < n,$$

and

$$\hat{A}^i \hat{b} = \begin{bmatrix} 0 \\ \sum_{j=0}^{k} F^{i-k-1+j} GCA^{k-j} b_1 \end{bmatrix}, \quad n \leqslant i \leqslant n + \nu - 1.$$

By changing the basis of the output space if necessary, let us write

$$CA^k b_1 = \begin{bmatrix} a \\ \vdots \end{bmatrix}, \quad a \neq 0.$$

We now choose

$$G = \begin{bmatrix} 1/a & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 0 & 0 & \cdots & 0 & \delta \\ \delta & 0 & \cdots & 0 & 0 \\ 0 & \delta & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \delta & 0 \end{bmatrix}.$$

We obtain

$$F^{i-k-1} GCA^k b_1 = \delta^{i-k-1} e_l,$$

where $1 \leqslant l \leqslant \nu$ is the integer such that $l = i - k \bmod \nu$, and e_l is the lth standard basis of $\mathbb{R}^\nu$. So we infer that

$$\left\{ \sum_{j=0}^{k} F^{i-k-1+j} GCA^{k-j} b_1 \,\middle|\, i = n, \ldots, n + \nu_1 \right\}$$
$$= \left\{ \delta^{i-k-1}(e_l + O(\delta)) \,\middle|\, i = n, \ldots, n + \nu_1 \right\}$$

are linearly independent for small δ, i.e., the pair $(\hat{A}, \hat{b})$ is controllable. Therefore, we conclude that system (4.2) is controllable and observable for the generic triplet (F, G, h). □

Lemma 4.2. *Let $Q(s)$ be an $(m + p) \times p$ minimal polynomial matrix of column degrees $\mu_1 \leqslant \cdots \leqslant \mu_p$. Then for $r < p$ and for the generic minimal $r \times (m + p)$ matrix $P(s)$ of row degrees $\nu_1 \geqslant \cdots \geqslant \nu_r$, $P(s)Q(s)$ is irreducible with McMillan degree*

$$\mu_{p-r+1} + \cdots + \mu_p + \nu_1 + \cdots + \nu_r.$$

The same is also true for the generic minimal $Q(s)$ if a minimal $P(s)$ is given.

Proof. A full size minor of $P(s)Q(s)$ is a sum of products of full size minors of $P(s)$ with $r \times r$ minors of $Q(s)$. Since the maximum degree of $r \times r$ minors of $Q(s)$ is $\mu_{p-r+1} + \cdots + \mu_p$, the maximum degree of full size minors of $P(s)Q(s)$ is $\mu_{p-r+1} + \cdots + \mu_p + \nu_1 + \cdots + \nu_r$ either for a fixed $Q(s)$ and the generic $P(s)$, or for a fixed $P(s)$ and the generic $Q(s)$.

Let $R(s) = Q^{\perp}$. It follows from Lemma 4.1 that

$$\begin{bmatrix} P(s) \\ R(s) \end{bmatrix} \tag{4.3}$$

is minimal either for a fixed $Q(s)$ and the generic $P(s)$, or for a fixed $P(s)$ and the generic $Q(s)$. Let $[Q(s), U(s)]$ be unimodular such that $R(s)U(s) = I_m$ (see Proposition 2.7). Then the matrix

$$\begin{bmatrix} P(s) \\ R(s) \end{bmatrix} \begin{bmatrix} Q(s), T(s) \end{bmatrix} = \begin{bmatrix} P(s)Q(s) & * \\ 0 & I_m \end{bmatrix}$$

has full rank $m + r$ for all s, which means that the full size minors of $P(s)Q(s)$ are relatively prime. □

We are now ready to obtain an improved estimate for the degrees of pole assigning compensators for the generic r-tuple of systems when $r \leqslant \max(m, p)$.

Theorem 4.3. *Let* $r \leqslant \max(m, p)$, *and q be the smallest integer satisfying the inequality*

$$q + \left(\lfloor q/\min(m, p)\rfloor + 1\right)(\max(m, p) - r) \geqslant \sum_{i=1}^{r} \lfloor n_i/\min(m, p)\rfloor. \tag{4.4}$$

Then a generic r-tuple of m-input, p-output systems of McMillan degree n_i, $i = 1, \ldots, r$, *respectively, can be arbitrarily pole assigned by a compensator of degree at most q.*

Proof. Without any loss of generality we assume $p \leqslant m$. The systems in $K_{p,m}^{n_i}$ with Forney indices

$$\mu_i := (\underbrace{k_i + 1, \ldots, k_i + 1}_{d_i}, \underbrace{k_i, \ldots, k_i}_{p-d_i}), \quad k_i = \lfloor n_i/p \rfloor, \ d_i = n_i - k_i p \tag{4.5}$$

form a nonempty Zariski open set. Each such system has a kernel representation $P_i(s)$ with row degrees μ_i. The set of all r-tuple of polynomial matrices of row degrees at most μ_i, $i = 1, \ldots, r$, is an affine space $\mathbb{A}$. The simultaneous dependent compensator constructed as in Remark 3.7 is a rational function of points in $\mathbb{A}$, and therefore the condition of Proposition 3.12 defines a Zariski open set on $\mathbb{A}$. This in turn defines a Zariski open set of $K_{p,m}^{n_1} \times \cdots \times K_{p,m}^{n_r}$. To finish the proof, we only need to show that this open set is nonempty; i.e., we need to construct a r-tuple of minimal polynomial matrices $(P_1(s), \ldots, P_r(s))$ such that:

1. Each $P_r(s)$ has row degrees μ_i as defined by (4.5).
2. If $\alpha_i(s)$ are the last rows of $P_i(s)$, then

$$\alpha := \begin{bmatrix} \alpha_1(s) \\ \vdots \\ \alpha_r(s) \end{bmatrix} \tag{4.6}$$

is minimal and has the generic dual indices

$$(\nu_1, \ldots, \nu_{m+p-r}) := (\underbrace{l, \ldots, l}_{m+p-r-e}, \underbrace{l+1, \ldots, l+1}_{e}), \tag{4.7}$$

where

$$l = \left\lfloor \frac{k_1 + \cdots + k_r}{m+p-r} \right\rfloor$$

and

$$e = k_1 + \cdots + k_r - l(m+p-r).$$

3. $d\chi_Q$ is onto, where Q is formed by the p columns of the lowest degrees of $\alpha^\perp$.

We first derive a simple formulation of $d\chi_Q$. Since $\alpha_i(s)Q(s) = 0$, it follows from Proposition 3.13 that

$$d\chi_Q(X(s)) = \big(\alpha_1(s)X(s)\eta_1(s), \ldots, \alpha_r(s)X(s)\eta_r(s)\big),$$

where $\eta_i(s)$ is the only nonzero column of adj $(P_i(s)Q(s))$. If we write

$$P_i(s) = \begin{bmatrix} \hat{P}_i(s) \\ \alpha_i(s) \end{bmatrix},$$

then the Jacobian becomes

$$d\chi_Q(X(s)) = \left(\det \begin{bmatrix} \hat{P}_1(s)Q(s) \\ \alpha_1(s)X(s) \end{bmatrix}, \ldots, \det \begin{bmatrix} \hat{P}_r(s)Q(s) \\ \alpha_r(s)X(s) \end{bmatrix}\right).$$

Based on such formulation of $d\chi_Q$, we construct $(P_1(s), \ldots, P_r(s))$ in two steps:

1. Choose $\alpha_i(s)$ such that ϕ defined by

$$\begin{bmatrix} z_1(s) \\ \vdots \\ z_r(s) \end{bmatrix} = \phi(X(s)) := \alpha(s)X(s)$$

is onto the space of all $r \times p$ polynomial matrices whose *ij*th entry has degree at most $k_i + \nu_j$, where ν_j is defined through (4.7).

2. Choose $\hat{P}_i(s)$ such that each ψ_i defined by

$$\psi_i(z_i(s)) := z_i(s)\eta_i(s) = \det \begin{bmatrix} \hat{P}_i(s)Q(s) \\ z_i(s) \end{bmatrix}$$

is onto the space of all polynomials of degree at most $n_i + q$.

Choose an α of row degrees $k_i = \lfloor n_i/p \rfloor,\ i = 1, \ldots, r$, such that it is minimal, and has the generic dual indices (4.7). Let

$$\alpha^\perp = \big[\beta_1(s), \ldots, \beta_{m+p-r}(s)\big]$$

be of column degrees $(\nu_1, \ldots, \nu_{m+p-r})$, and define

$$Q(s) = \big[\beta_1(s), \ldots, \beta_p(s)\big].$$

Then the degree q of $Q(s)$ is the smallest integer satisfying (3.2) (see the proof of Theorem 3.2 in [9]) and

$$l = \nu_1 = \lfloor q/p \rfloor.$$

For $X(s) = [x_1(s), \dots, x_p(s)] \in \mathscr{M}_p^q$, where $\mathscr{M}_p^q$ is defined by (3.4), define linear maps $\phi_i : \mathbb{R}^{(m+p)(\nu_i+1)} \to \mathbb{R}^{r(\nu_i+1)+\sum_{j=1}^r k_j}$:

$$\phi_i(x_i) = \alpha(s)x_i(s), \quad i = 1, \dots, p.$$

By [1] ϕ_i has a rank (over $\mathbb{R}$)

$$(m+p)(\nu_i+1) - \sum_{\nu_j \leqslant \nu_i} (\nu_i + 1 - \nu_j).$$

Note that when $\nu_i = l$

$$\sum_{\nu_j \leqslant \nu_i} (\nu_i + 1 - \nu_j) = m + p - r - e$$

and

$$\begin{aligned} \operatorname{rank} \phi_i &= (m+p)(l+1) - m - p + r + e \\ &= (l+1)r + l(m+p-r) + e \\ &= (\nu_i+1)r + \sum_{j=1}^r k_j. \end{aligned}$$

When $\nu_i = l+1$, we obtain

$$\sum_{\nu_j \leqslant \nu_i} (\nu_i + 1 - \nu_j) = 2(m + p - r) - e$$

and

$$\begin{aligned} \operatorname{rank} \phi_i &= (m+p)(l+2) - 2(m+p-r) + e \\ &= (l+2)r + l(m+p-r) + e \\ &= (\nu_i+1)r + \sum_{j=1}^r k_j. \end{aligned}$$

In either case ϕ_i is onto the space of all column r-vectors whose degree of the jth entry is at most $k_j + \nu_i$ $(= \mathbb{R}^{r(\nu_i+1)+\sum_{j=1}^r k_j})$. So $\phi(X(s)) = (\phi_1(x_1), \dots, \phi_p(x_p))$ is onto.

Next we choose a minimal $\hat{P}_i(s)$ of row degrees

$$(\underbrace{k_i+1, \dots, k_i+1}_{d_i}, \underbrace{k_i, \dots, k_i}_{p-1-d_i}) \tag{4.8}$$

such that $\hat{P}_i(s)Q(s)$ is irreducible with McMillan degree $n_i + q - \lfloor n_i/p \rfloor - \lfloor q/p \rfloor$ (see Lemma 4.2). Note that $\hat{P}_h Q_h$ has full rank, because its full size minors are the coefficients of the monomials of $s^{n_i+q-\lfloor n_i/p\rfloor-\lfloor q/p\rfloor}$ of the corresponding full size minors of $\hat{P}_i(s)Q(s)$, and therefore not all of them are zero. Let us define

$$\mathscr{Z}_i = \left\{z_i(s) = (z_{i1}(s), \ldots, z_{ip}(s)) \,\middle|\, \deg z_{ij}(s) \leqslant k_i + \nu_j\right\}.$$

We show that the linear map $\psi_i : \mathscr{Z}_i \to \mathbb{R}^{n_i+q+1}$ defined by

$$\psi_i(z_i(s)) = \det\begin{bmatrix} \hat{P}_i(s)Q(s) \\ z_i(s)\end{bmatrix}$$

is onto the space of all polynomials of degree $\leqslant n_i + q$. Note that $\mathscr{Z}_i = \mathbb{R}^{p(k_i+1)+q}$. So we need to show that

$$\dim\ker\psi_i = p - 1 - d_i.$$

If $z_i(s) \in \ker\psi_i$, then by Proposition 2.4

$$z_i(s) = \left[a_1(s), \ldots, a_{p-1}(s)\right]\hat{P}_i(s)Q(s)$$

for some polynomials $\{a_j(s)\}$. We claim that $a_j(s) = 0$ for $j \leqslant d_i$, and $a_j(s) = a_j$ for $j > d_i$. If not, then

$$\deg y(s) := \deg\left[a_1(s), \ldots, a_{p-1}(s)\right]\hat{P}_i(s) > k_i.$$

Since $y_h \in$ row space$[\hat{P}_i]_h$, one must have $y_hQ_h \neq 0$. Assume that the jth entry of y_hQ_h is nonzero. Then $\deg z_{ij}(s) > k_i + \nu_j$ and $z_i(s) \notin \mathscr{Z}_i$. Therefore

$$\ker\psi_i = \left\{\left[0, \ldots, 0, a_{d_i+1}, \ldots, a_{p-1}\right]\hat{P}_i(s)Q(s)\right\}$$

and $\dim\ker\psi_i = p - 1 - d_i$. □

Theorem 4.3 improves the results of [6,8]. In particular, when $\min(m, p) = 1$, inequality (4.4) reduces to

$$q(\max(m, p) + 1 - r) + \max(m, p) - r \geqslant \sum_{i=1}^{r} n_i, \tag{4.9}$$

which is precisely the inequality obtained in [8]. On the other hand when $r = \max(m, p)$, the smallest degree of the compensator which simultaneously pole assigns a $\max(m, p)$ plants generically is given by

$$\sum_{i=1}^{r}\left\lfloor\frac{n_i}{\min(m, p)}\right\rfloor,$$

which should be compared with the smallest degree obtained in [8] given by $\sum_{i=1}^{r} n_i$. Thus, Theorem 4.3 improves the result derived by Ghosh and Byrnes [6].

In our next result we show that a generic r-tuple of systems is simultaneously pole assignable when $r > m + p$ if the McMillan degrees of at least $\min(m, p)$ systems are not “too different”. It also gives an estimate for the degrees of pole assigning compensators for the generic r-tuple of systems when $\max(m, p) < r < m + p$.

Theorem 4.4. *If* $\max(m, p) < r < m + p$ *and* (*re-label the plants if necessary*)

$$\left| \left\lfloor \frac{n_j + N}{m + p - r} \right\rfloor - \left\lfloor \frac{n_k + N}{m + p - r} \right\rfloor \right| \leqslant 1 \quad \textit{for } 1 \leqslant j < k \leqslant \min(m, p), \quad (4.10)$$

where

$$N = \sum_{i=\min(m,p)+1}^{r} \left\lfloor \frac{n_i}{\min(m, p)} \right\rfloor,$$

then the generic r-tuple of m-input systems of degrees n_i, $i = 1, \ldots, r$, *respectively, can be arbitrarily pole assigned by a compensator of degree less than or equal to*

$$q = \sum_{i=1}^{\min(m,p)} \left\lfloor \frac{n_i + N}{m + p - r} \right\rfloor.$$

Proof. Without loss of generality, we assume that $p \leqslant m$. By Propositions 2.6 and 3.12, and Remark 3.9, such systems certainly form a Zariski open subset. So we only need to show that it is nonempty; i.e., we need to construct an r-tuple of minimal polynomial matrices of row degrees μ_i, $i = 1, \ldots, r$ (as defined by (4.5)), such that the Jacobian $d\chi_Q$ described in Proposition 3.12 is onto for the simultaneous dependent compensator $Q(s)$ constructed as in Remark 3.9.

Let $Q(s)$ be constructed as in Remark 3.9. Then for $i = 1, \ldots, p$, the ith column of $P_i(s)Q(s)$ is zero, and for $i = p + 1, \ldots, r$, the last row of $P_i(s)Q(s)$ is zero. Therefore, $d\chi_Q(X) = (d\chi_1(X), \ldots, d\chi_r(X))$ has the form

$$d\chi_i(X) = (-1)^{p-i} \det\left[P_i(s)\hat{Q}_i(s), P_i(s)x_i(s)\right] \quad \text{for } 1 \leqslant i \leqslant p,$$

and

$$d\chi_i(X) = \det \begin{bmatrix} \hat{P}_i(s)Q(s) \\ \alpha_i(s)X(s) \end{bmatrix} \quad \text{for } p < i \leqslant r,$$

where $\hat{Q}_i(s)$ is the $(m + p) \times (p - 1)$ sub-matrix of $Q(s)$ formed by removing the ith column of $Q(s)$, and $\hat{P}_i(s)$ is the $(p - 1) \times (m + p)$ sub-matrix of $P(s)$ consisting of the first $p - 1$ rows of $P_i(s)$.

We construct the r-tuple of systems in three steps:

1. Choose minimal

$$\alpha := \begin{bmatrix} \alpha_{p+1}(s) \\ \vdots \\ \alpha_r(s) \end{bmatrix}$$

of row degrees $\{\lfloor n_{p+1}/p \rfloor, \ldots, \lfloor n_r/p \rfloor\}$ such that they have the dual Forney indices $\nu = (\nu_1, \ldots, \nu_{m+2p-r})$

$$\nu_i = \begin{cases} l & \text{for } 1 \leqslant i \leqslant m + 2p - r - e, \\ l + 1 & \text{for } m + 2p - r - e < i \leqslant m + 2p - r, \end{cases}$$

where

$$l=\left\lfloor \frac{N}{m+2p-r} \right\rfloor, \quad N=\sum_{i=p+1}^{r} \left\lfloor \frac{n_i}{p} \right\rfloor \quad \text{and} \quad e=N-l(m+2p-r).$$

2. Choose minimal $P_i(s),\ i=1,\dots,p$, of row degrees μ_i, and consequently determine $Q(s)$ such that the corresponding

$$d\chi_i(X)=(-1)^{p-i}\det\left[P_i(s)\hat{Q}_i(s),\ P_i(s)x_i(s)\right]$$

is onto if $\{x_i\}$ are restricted to the column space of $\alpha^{\perp}$.

3. Choose $(p-1)\times(m+p)$ minimal $\hat{P}_i(s)$ of row degrees

$$\mu_i := (\underbrace{k_i+1,\dots,k_i+1}_{d_i}, \underbrace{k_i,\dots,k_i}_{p-d_i-1}), \quad k_i=\lfloor n_i/p \rfloor, \quad d_i=n_i-k_i p$$

for $i=p+1,\dots,r$ such that $\hat{P}_i(s)Q(s)$ is irreducible with McMillan degree $n_i+q-\lfloor n_i/p\rfloor-\lfloor q/p\rfloor$, and define

$$P_i(s)=\begin{bmatrix}\hat{P}_i(s)\\ \alpha_i(s)\end{bmatrix}, \quad i=p+1,\dots,r.$$

The existence of such α_i and $\hat{P}_i(s),\ i=p+1,\dots,r$, and surjectivity of the corresponding $d\chi_i$ are proved in the proof of Theorem 4.3. So we only need to show existence of the $P_i(s)$ in the second step.

Choose $P_i(s),\ i=1,\dots,p$, such that

(1)

$$\bar{P}_i(s) := \begin{bmatrix}P_i(s)\\ \alpha(s)\end{bmatrix} \tag{4.11}$$

is minimal with dual Forney indices

$$(\underbrace{\rho_i,\dots,\rho_i}_{m+p-r-\delta_i}, \underbrace{\rho_i+1,\dots,\rho_i+1}_{\delta_i}),$$

$$\rho_i=\left\lfloor \frac{n_i+N}{m+p-r} \right\rfloor, \quad \delta_i=n_i+N-\rho_i(m+p-r). \tag{4.12}$$

(2) $Q(s)=[q_1(s),\dots,q_p(s)]$ is minimal, where $q_i(s)$ is the first column of $\bar{P}_i^{\perp}(s)$ of column degrees (4.12).

(3) $P_i(s)\hat{Q}_i(s)$ is irreducible with McMillan degree

$$n_i+q-\lfloor n_i/p\rfloor-\rho_i.$$

Certainly the generic $P_i(s)$ satisfies (1). The set of all $\{P_i(s)\}$ satisfying (2) is Zariski open, and one can always start from $Q(s)$ to construct the corresponding $\{P_i(s)\}$. So the generic $\{P_i(s)\}$ also satisfies (2). By Lemma 4.2 for a fixed $\hat{Q}_i(s)$, the generic $P_i(s)$ satisfies (3) (note that $\hat{Q}_i(s)$ is independent of $P_i(s)$). Therefore, we can start from a p-tuple $\{P_i(s)\}$ satisfying (1) and (2), and make a small perturbation so that (3) are satisfied by all $P_i(s)$.

Let

$$\mathscr{L}_1 := \left\{ x(s) \in \text{col. span}\, \alpha^{\perp} \,|\, \deg x(s) \leqslant \rho_i \right\}.$$

We now show that for such $P_i(s)$,

$$d\chi_i(X) = (-1)^{p-i} \det\left[P_i(s)\hat{Q}_i(s), P_i(s)x_i(s) \right], \quad x(s) \in \mathscr{L}_1,$$

is onto the space of all polynomial of degree $\leqslant n_i + q$, where

$$q = \sum_{j=1}^{p} \rho_i.$$

We compute the dimension of $\ker d\chi_i$ over $\mathbb{R}$. As in the proof of Theorem 4.3, an $x(s)$ is in the $\ker d\chi_i$ if, and only if, $P_i(s)x(s) = P_i(s)\hat{Q}_i(s)a(s)$ for some polynomial column vector $a(s)$ such that

$$\deg \hat{Q}_i(s)a(s) \leqslant \rho_i,$$

i.e.,

$$x(s) = \hat{Q}_i(s)a(s) + y(s)$$

for some $y(s)$ of degree $\leqslant \rho_i$ in the column space of $\bar{P}_i^{\perp}(s)$. So

$$\dim \ker d\chi_i = \dim \mathscr{L}_2 + \dim \mathscr{L}_3,$$

where

$$\mathscr{L}_2 := \left\{ x(s) \in \text{col. span}\, \hat{Q}_i(s) \mid \deg x(s) \leqslant \rho_i \right\}$$

and

$$\mathscr{L}_3 := \left\{ x(s) \in \text{col. span}\, \bar{P}_i^{\perp}(s) \mid \deg x(s) \leqslant \rho_i \right\}.$$

By the assumption of the theorem, we have $|\rho_i - \rho_j| \leqslant 1$. So by Proposition 2.4 we write

$$\begin{aligned}\dim \mathscr{L}_2 &= \sum_{j \neq i} (\rho_i - \rho_j + 1) \\ &= p\rho_i - q + p - 1.\end{aligned}$$

Similarly, since $\bar{P}_i^{\perp}(s)$ has column degrees (4.12)

$$\dim \mathscr{L}_3 = (m + p - r)(\rho_i + 1) - n_i - N.$$

Therefore,

$$\dim \ker d\chi_i = (m + 2p - r)(\rho_i + 1) - q - n_i - 1 - N.$$

The dimension of $\mathscr{L}_1$ (over $\mathbb{R}$) is given by

$$\begin{aligned}\dim \mathscr{L}_1 &= \sum_{j=1}^{m+2p-r} (\rho_i - \nu_j + 1) \\ &= (m + 2p - r)(\rho_i + 1) - N.\end{aligned}$$

Therefore,

$$\dim \mathscr{L}_1 - \dim \ker d\chi_i = n_i + q + 1$$

and $d\chi_i$ is onto. □

5. Conclusion

This paper settles an outstanding open problem initiated by Saeks and Murray [15] and by Vidyasagar and Viswanadham [17].

Problem 5.1. How many linear time invariant m-input p-output plants of degrees n_i, $i = 1, \dots, r$, can be simultaneously stabilized or simultaneously pole assigned generically by a linear, time invariant, nonswitching, dynamic compensator?

"Less than the sum of the number of inputs and outputs"

is the answer to the above problem provided by this paper. Moreover, in this case, there is always an upper bound on the degree of the simultaneously pole assigning compensator.

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