Geometry and Control of Human Eye Movements

Ashoka D. Polpitiya, *Member, IEEE*, Wijesuriya P. Dayawansa, *Fellow, IEEE*, Clyde F. Martin, *Fellow, IEEE*, and Bijoy K. Ghosh, *Fellow, IEEE*

***Abstract*—In this paper, we study the human oculomotor system as a simple mechanical control system. It is a well known physiological fact that all eye movements obey Listing's law, which states that eye orientations form a subset consisting of rotation matrices for which the axes are orthogonal to the normal gaze direction. First, we discuss the geometry of this restricted configuration space (referred to as the Listing space). Then we formulate the system as a simple mechanical control system with a holonomic constraint. We propose a realistic model with musculotendon complexes and address the question of controlling the gaze. As an example, an optimal energy control problem is formulated and numerically solved.**

***Index Terms*—Eye movements, geodesics, Hill model, Listing's law, simple mechanical control systems.**

I. Introduction

BIOLOGICAL systems are becoming more appealing to approaches that are commonly used in systems theory and suggest new design principles that may have important practical applications in manmade systems. The principles of control theory are central to many of the key questions in biological engineering. Eye movements, for an example, reflect how the brain and the musculotendon system work in unison to control the gaze directions while ensuring that attitudes are confined to a certain subset so as to avoid entanglements of blood vessels, nerve fibers etc.

Modeling the eye plant in order to generate various eye movements both normal and symptomatic, has been one of the important goals among neurologists, physiologists, and engineers for a long time. Since as early as 1845 (e.g., work of Listing, Donders, Helmholtz etc.), physiologists and engineers have created models in order to help understand various eye movements (see [1]). The precise coordination in muscles when the eye is rotated by the action of six extra-ocular muscles (EOMs) (see Fig. 1), has been an important topic in treating various ocular disorders. The eyes rotate with three degrees of freedom, making it an interesting yet simpler problem compared to other complex human movement systems. By comparison, 24 muscles and tendons have to be taken into consideration for a two dimensional simplification of human walking [2].

Previous studies which used modeling as a means of understanding the control of eye movements have adopted two main approaches. One focusing on the details of the properties of the EOMs (i.e., biomechanically "correct") [3], [4] and the other focusing on control mechanisms using oversimplified linear models with all the details of the above EOM properties ignored but focusing on the information processing and control aspects [5], [6].

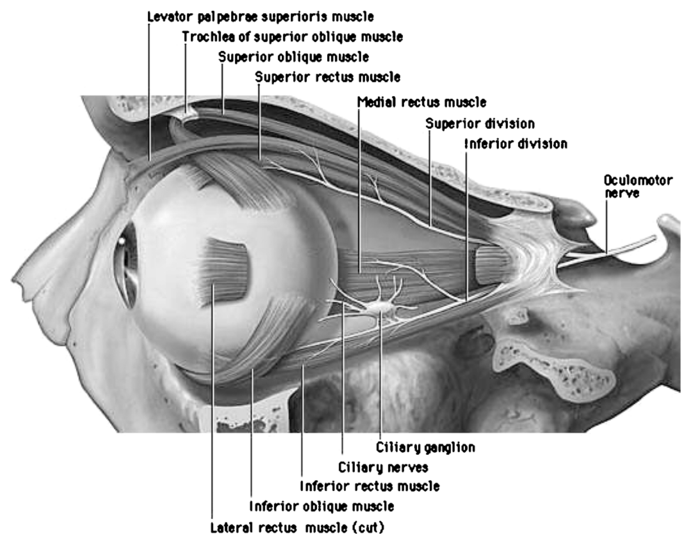

Fig. 1. Anatomy of the eye (courtesy of Yale University School of Medicine).

In spite of several notable studies of three dimensional eye movements, there has not been a rigorous treatment of the topic in the framework of modern control theory and geometric mechanics. Assuming the eye to be a rigid sphere, the problem can be treated as a mechanical control system and the results in classical mechanics and modern nonlinear control systems can be promptly applied. The geometric structure of mechanical systems, in general, gives way to stronger control algorithms than those obtained for generic nonlinear systems [7]. Thus the approach proposed here takes the advantage of richly developed disciplines of mechanics as well as control theory. In the area of mechanics, unlike the classic approach by [8], recent works in [9]–[13] and [14] develop a geometric theory with Lagrangian and Hamiltonian viewpoints. On the other hand control theory consists of a large as well as elegant collection of literature and specially beginning late 1970s, the works of [15], [16], and [17], etc., have introduced geometric tools to nonlinear control problems.

The discussion in this paper may be viewed as an attempt to study and model the eye movement system as a "*simple mechanical control system*" [9]. Such a model in isolation may only be an academic exercise. However, it can have both clinical utility in treating ocular disorders, and scientific importance in understanding the human movement system in general.

From a functional viewpoint only the rotational aspects of the eye movement are interesting, hence it is natural to start with $\mathsf{SO}(3)$, the space of 3×3 rotation matrices, as its configuration space. However, from a physiological viewpoint, only the

Manuscript received June 1, 2005; revised April 21, 2006 and May 30, 2006.
A. D. Polpitiya is with the Pacific Northwest National Laboratory, Richland, WA 99352 USA (e-mail: ashoka.polpitiya@pnl.gov).
W. P. Dayawansa, C. F. Martin, and B. K. Ghosh are with Texas Tech University, Lubbock, TX 79409 USA.
Digital Object Identifier 10.1109/TAC.2006.887902

0018-9286/$25.00 © 2007 IEEE

gaze direction vector is of primary importance, and the orientation of the eye is otherwise secondary. From simple geometric reasoning it follows that each gaze direction of the eye corresponds to a circle of rotation matrices in the configuration space. Thus, there is an ambiguity as to which rotation matrix is to be employed to produce a particular gaze direction. Listing's law describes precisely how this ambiguity is resolved: all rotation matrices employed have their axes of rotation orthogonal to the standard (or frontal) gaze direction [18]. Thus, the dynamics of the eye may be treated as a mechanical system with holonomic constraints, which in essence limit the configuration space to be a two dimensional submanifold of $\mathsf{SO}(3)$. We will refer to it as the ***Listing Space***. We will first describe basic geometric features of the Listing Space. This will enable us to formulate dynamic equations of the eye motion using various neuro/muscular models. We make the following fundamental assumptions throughout this paper.

- The eye is a perfect sphere.
- All eye movements obey Listings law.

We remark here that the second assumption pertains to all eye configurations throughout its motions, and not just on the initial and final points of an eye movement. Also the first assumption may be removed completely at the expense of a slightly more complicated system of equations.

There have been several notable studies on the geometry of eye rotations in the past (see e.g., [19]–[22]). In particular, [19] describes this geometry using Lie theory, as the quotient space $\mathsf{SO}(3)/\mathsf{SO}(2)$. We point out here that the Listing space isn't actually diffeomorphic to $\mathsf{SO}(3)/\mathsf{SO}(2)$, but in fact is equal to a submanifold of $\mathsf{SO}(3)$ which is diffeomorphic to the two dimensional real projective space. As far as we are aware our study is the first to explicitly describe the Riemannian geometry of the submanifold of Listing rotations precisely. This then will enable one to formulate dynamic equations of the eye movements using various neuro/muscular models.

II. Notation and Terminology

Let us begin with the space of quaternions (see [23]) denoted by $\mathbf{Q}$. We write each $a \in \mathbf{Q}$ as $a_0\overrightarrow{\mathbf{1}} + a_1\overrightarrow{\mathbf{i}} + a_2\overrightarrow{\mathbf{j}} + a_3\overrightarrow{\mathbf{k}}$, call $a_1\overrightarrow{\mathbf{i}} + a_2\overrightarrow{\mathbf{j}} + a_3\overrightarrow{\mathbf{k}}$ its vector part, and $a_0\overrightarrow{\mathbf{1}}$ its scalar part. The vector $a_1\overrightarrow{\mathbf{i}} + a_2\overrightarrow{\mathbf{j}} + a_3\overrightarrow{\mathbf{k}}$ will be identified with $(a_1, a_2, a_3) \in \mathbb{R}^3$ without any explicit mention of it. When there is no confusion we drop $\overrightarrow{\mathbf{1}}$ from the scalar part, and simply write it as a_0. The vector part of a quaternion a will be denoted by $\mathbf{vec}(a)$, or simply by $\mathbf{a}$, and the scalar part will be denoted by $\mathbf{scal}(a)$. Thus, we have the two maps

$$\mathbf{vec} : \mathbf{Q} \to \mathbb{R}^3, \quad a \mapsto (a_1, a_2, a_3)$$

and

$$\mathbf{scal} : \mathbf{Q} \to \mathbb{R}, \quad a \mapsto a_0.$$

Space of unit quaternions will be identified with the unit sphere in $\mathbb{R}^4$, and denoted by S^3. Each $q \in S^3$ can be written as $q = \cos(\alpha/2)\overrightarrow{\mathbf{1}} + \sin(\alpha/2)n_1\overrightarrow{\mathbf{i}} + \sin(\alpha/2)n_2\overrightarrow{\mathbf{j}} + \sin(\alpha/2)n_3\overrightarrow{\mathbf{k}}$, where, $\alpha \in [0, \pi]$ and (n_1, n_2, n_3) is a unit vector in $\mathbb{R}^3$. We denote by $\mathbf{rot}$ the standard map from S^3 into $\mathsf{SO}(3)$ which maps $\cos(\alpha/2)\overrightarrow{\mathbf{1}} + \sin(\alpha/2)n_1\overrightarrow{\mathbf{i}} + \sin(\alpha/2)n_2\overrightarrow{\mathbf{j}} + \sin(\alpha/2)n_3\overrightarrow{\mathbf{k}}$ to a rotation around the axis n by a counterclockwise angle α. There are two explicit ways of describing this map. First, it is easy to verify that

$$\mathbf{rot}(\mathrm{q})(v_1, v_2, v_3) = \mathbf{vec}(q.(v_1\overrightarrow{\mathbf{i}} + v_2\overrightarrow{\mathbf{j}} + v_3\overrightarrow{\mathbf{k}}).q^{-1}).$$

Second, we have the equation shown at the bottom of the page.

III. Listing Manifold Is Diffeomorphic to the Projective Space

Listings law states that all eye rotations have an axis orthogonal to the primary gaze direction. If we were to take the (x_1, x_2, x_3) axes such that x_3 axis is aligned with the normal gaze direction, then Listing's law amounts to a statement that only eye rotations allowed by the physiology are those that fix an axis orthogonal to $[0, 0, 1]'$, i.e., that all eye rotations have quaternion representations $q \in S^3$ with $q_3 = 0$. We denote by ***List***, the subset of $\mathsf{SO}(3)$ which obey Listing's law. More specifically

$$\boldsymbol{List} = \left\{ R \in \mathsf{SO}(3) \mid \exists v = [v_1, v_2] \in \mathbb{R}^2 \backslash \{0\} \ni R \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \right\}.$$

In this section, we will show that ***List*** is diffeomorphic to the projective space $\mathbb{RP}^2$. Let us now consider the map

$$\begin{aligned} &\mathbf{emb} : \mathbb{R}^3 \to S^3 \\ &(x_1, x_2, x_3) \mapsto [1 \quad x_1 \quad x_2 \quad x_3]^T [1 + \|x\|^2]^{-\frac{1}{2}} \end{aligned}$$

where we note that $\mathbf{emb}(x)$ is the quaternion which describes a rotation around $(1/\|x\|)x$ by an angle $2\arctan(\|x\|)$ (where the angle is chosen to be in the interval $[0, \pi)$). The ambiguity at $x = 0$ is resolved by mapping it to $\overrightarrow{\mathbf{1}}$. Therefore, each vector with zero x_3 coordinate describes a unique Listing rotation. However, those Listing rotations with an angle of rotation equal to π are missing here. Let us observe that a rotation by π around an axis n is identical to a rotation by an

$$\mathbf{rot}(\mathrm{q}) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix}$$

angle $-\pi$ around $-n$. Thus we may describe ***List*** by appropriately compactifying $\mathbb{R}^3$. This compactification is best understood in the following way. Let us start with $\mathbb{R}^4$ with coordinates (x_0, x_1, x_2, x_3) and consider the usual projective equivalence relation that would collapse one dimensional subspaces to points. This way, each $(x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with the equivalence class of $(1, x_1, x_2, x_3)$, hence associated with the unique rotation of $\mathbf{emb}(x_1, x_2, x_3)$. Equivalence classes of $(0, x_1, x_2, x_3)$ are uniquely associated with rotations by an angle of π or $-\pi$. This association is unambiguous since rotation by π and $-\pi$ around the unit vector n amounts to the same rotation matrix in $\mathsf{SO}(3)$. Thus we conclude that the space ***List*** is diffeomorphic to $\mathbb{RP}^2$.

This identification provides an obvious way to come up with local coordinates on ***List***. However, it turns out that the description of a natural Riemannian metric on ***List*** is quite awkward using these coordinates. Hence, we will use an axis-angle local coordinate system to carry out most of our computation. The two local coordinates (θ, ϕ) describe the polar coordinate angle of the axis of rotation in the (x_1, x_2) plane and the angle of rotation around the axis, respectively. Here, we take $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$. Of course, we must keep in mind that these fail to be local coordinates when $\phi = 0$ or $\phi = 2\pi$ since in these cases the corresponding rotation is the identity regardless of the value of θ.

IV. Riemannian Metric on *List*

We have seen in the previous section that eye rotations are confined to a submanifold of $\mathsf{SO}(3)$. In order to write down equations of motion, one needs to know the kinetic and potential energies of the eye in motion. The former is given by the induced Riemannian metric on the Listing submanifold, induced by the Riemannian metric on $\mathsf{SO}(3)$ which corresponds to the moment of inertia of the eye ball. We carry out the computation of this induced metric in this section.

Here, we assume that the eye is a perfect sphere, and its inertia tensor is equal to $\mathbb{I}_{3\times 3}$. This is associated with the left invariant Riemannian metric on $\mathsf{SO}(3)$ given by

$$\langle \mathbf{\Omega}(e_i), \mathbf{\Omega}(e_j) \rangle_{\mathbb{I}} = \delta_{i,j}$$

where

$$\mathbf{\Omega}(e_k) = \begin{bmatrix} 0 & \delta_{3,k} & -\delta_{2,k} \\ -\delta_{3,k} & 0 & \delta_{1,k} \\ \delta_{2,k} & -\delta_{1,k} & 0 \end{bmatrix}$$

and $\{\delta_{l,m}\}$ denotes the Kronecker delta function. An easy way to carry out computation using this Riemannian metric is provided by the isometric submersion $\mathbf{rot}$. Notice that $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$ is an orthonormal basis of $\mathsf{T}_{\vec{\mathbf{1}}} S^3$, and

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ \sin(t/2) \\ 0 \\ 0 \end{bmatrix}\right) = e^{t\mathbf{\Omega}(e_1)}$$

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ \sin(t/2) \\ 0 \end{bmatrix}\right) = e^{t\mathbf{\Omega}(e_2)}$$

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ 0 \\ \sin(t/2) \end{bmatrix}\right) = e^{t\mathbf{\Omega}(e_3)}.$$

Hence, it follows that $\mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{i}} = 2\mathbf{\Omega}(e_1)$, $\mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{j}} = 2\mathbf{\Omega}(e_2)$ and $\mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{k}} = 2\mathbf{\Omega}(e_3)$, hence $\{\mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{i}}/2, \mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{j}}/2, \mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{k}}/2\}$ is an orthonormal frame in $\mathsf{T}_{\mathrm{Id}}(\mathsf{SO}(3))$. Now, since $\mathbf{rot}$ is equivariant under left translations, and Riemannian metric on $\mathsf{SO}(3)$ is left invariant, it follows that $\{\mathbf{rot}_{*\mathrm{q}}\vec{\mathbf{i}}/2, \mathbf{rot}_{*\mathrm{q}}\vec{\mathbf{j}}/2, \mathbf{rot}_{*\mathrm{q}}\vec{\mathbf{k}}/2\}$ is an orthonormal basis of $\mathsf{T}_{\mathbf{rot}(q)}\mathsf{SO}(3)$ for all $q \in S^3$.

$$\begin{array}{ccc} S^3 & \xrightarrow{\mathbf{rot}} & \mathsf{SO}(3) \\ \downarrow & & \downarrow \\ \mathsf{T}_q S^3 & \xrightarrow{\mathbf{rot}} & \mathsf{T}_{\mathbf{rot}(\mathrm{q})}\mathsf{SO}(3). \end{array}$$

Let us now use the orthonormal frame $\{q.\vec{\mathbf{i}}/2, q.\vec{\mathbf{j}}/2, q.\vec{\mathbf{k}}/2\}$ of $\mathsf{T}_q S^3$ to compute the Riemannian metric on ***List*** induced from $\mathsf{SO}(3)$. We define

$$g_{11} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle$$
$$g_{12} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} \right\rangle$$
$$g_{22} = \left\langle \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi} \right\rangle. \tag{1}$$

Let $\rho : [0, \pi] \times [0, 2\pi] \to S^3$,

$$\rho(\theta, \phi) = \begin{bmatrix} \cos(\phi/2) \\ \cos(\theta)\sin(\phi/2) \\ \sin(\theta)\sin(\phi/2) \\ 0 \end{bmatrix}.$$

This can be illustrated as follows:

$$\begin{array}{ccc} \mathbf{\mathit{List}} & \xrightarrow{\rho} & S^3 \\ \downarrow & & \downarrow \\ \mathsf{T}_{(\theta,\phi)}\mathbf{\mathit{List}} & \xrightarrow{\rho_*} & \mathsf{T}_{\rho(\theta,\phi)} S^3. \end{array}$$

Then

$$\mathcal{J}(\rho)(\theta, \phi) = \left(\rho_*\left(\frac{\partial}{\partial\theta}\right) \quad \rho_*\left(\frac{\partial}{\partial\phi}\right)\right)$$
$$= \begin{pmatrix} 0 & -\frac{1}{2}\sin(\phi/2) \\ -\sin(\theta)\sin(\phi/2) & \frac{1}{2}\cos(\theta)\cos(\phi/2) \\ \cos(\theta)\sin(\phi/2) & \frac{1}{2}\sin(\theta)\cos(\phi/2) \\ 0 & 0 \end{pmatrix}.$$

Using quaternion multiplication, we compute the following [23]:

$$\rho(\theta,\phi).\,\overrightarrow{\mathbf{i}} = \begin{bmatrix} -\cos(\theta)\sin(\phi/2) \\ \cos(\phi/2) \\ 0 \\ -\sin(\theta)\sin(\phi/2) \end{bmatrix}$$

$$\rho(\theta,\phi).\,\overrightarrow{\mathbf{j}} = \begin{bmatrix} -\sin(\theta)\sin(\phi/2) \\ 0 \\ \cos(\phi/2) \\ \cos(\theta)\sin(\phi/2) \end{bmatrix}$$

$$\rho(\theta,\phi).\,\overrightarrow{\mathbf{k}} = \begin{bmatrix} 0 \\ \sin(\theta)\sin(\phi/2) \\ -\cos(\theta)\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix}.$$

Then, for $\theta = 0$, it is easily observed that

$$\begin{aligned} \rho_{*(0,\phi)}(\frac{\partial}{\partial\theta}) &= \sin(\phi/2)\cos(\phi/2)\rho(0,\phi).\,\overrightarrow{\mathbf{j}} \\ &\quad - \sin^2(\phi/2)\rho(0,\phi).\,\overrightarrow{\mathbf{k}} \\ \rho_{*(0,\phi)}(\frac{\partial}{\partial\phi}) &= \frac{1}{2}\rho(0,\phi).\,\overrightarrow{\mathbf{i}}. \end{aligned}$$

Hence, we have

$$\langle u,v\rangle_{\boldsymbol{List}} = \langle \rho_{*(0,\phi)}(u), \rho_{*(0,\phi)}(v)\rangle_{S^3}, \quad u,v \in \mathsf{T}_{(0,\phi)}\boldsymbol{List}.$$

Therefore, using the relations given in (1) we obtain

$$\begin{aligned} g_{11} &= \sin^2(\phi/2) \\ g_{12} &= 0 \\ g_{22} &= \frac{1}{4}. \end{aligned}$$

Thus, the Riemannian metric on ***List*** is

$$g = \sin^2(\phi/2)d\theta^2 + \frac{1}{4}d\phi^2.$$

Notice that this expression is singular at $\phi = 0$. This represents the fact that (θ,ϕ) fail to be local coordinates around $\phi = 0$.

V. Geometry of the Listing Space

A. Connection on *List*

Now, we compute the Riemannian \index{connection}connection ∇, on ***List***. Even though one may carry out all derivations of equations without the need of explicit formulas of the associated Riemannian connection, this would leave the geometric description of the Listing space incomplete. Among other things, this description of the connection would enable a straightforward computation of the geodesics and the curvature of the Listing submanifold. It is well known that ∇ is uniquely defined by the formula (see [24])

$$\begin{aligned} 2\langle\nabla_X Y, Z\rangle = &\mathcal{L}_X\langle Y,Z\rangle + \mathcal{L}_Y\langle X,Z\rangle - \mathcal{L}_Z\langle X,Y\rangle \\ &-\langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle. \end{aligned}$$

Let us use subscripted coordinates (y_1, y_2) to denote (θ,ϕ). The Riemannian connection ∇ can be described in local coordinates (y_1,y_2) via

$$\nabla_{\partial y_i/\partial y_j} = \Gamma^k_{ij}\partial/\partial y_k$$

where Γ^k_{ij} are the Christoffel symbols [25]. They can be calculated using the following standard formula:

$$\Gamma^k_{ij} = \sum_{h=1}^{2}\frac{g^{ih}}{2}\left\{\frac{\partial g_{hj}}{\partial y_k} + \frac{\partial g_{hk}}{\partial y_j} - \frac{\partial g_{jk}}{\partial y_h}\right\}$$

where $i,j,k = 1,2$ and the associated symmetry properties. Here, we have

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \sin^2(\phi/2) & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$
and
$$(g^{ij}) = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \csc^2(\phi/2) & 0 \\ 0 & 4 \end{pmatrix}.$$

Thus, we obtain the following expressions for the Christoffel symbols:

$$\begin{aligned} &\Gamma^1_{11} = 0 \quad \Gamma^2_{11} = -\sin(\phi) \\ &\Gamma^1_{12} = \frac{1}{2\tan(\phi/2)} \quad \Gamma^1_{21} = \frac{1}{2\tan(\phi/2)} \\ &\Gamma^2_{12} = 0 \quad \Gamma^2_{21} = 0 \\ &\Gamma^1_{22} = 0 \quad \Gamma^2_{22} = 0. \end{aligned}$$

B. Equations of Geodesics on *List*

The Riemannian metric g computed in the Section IV gives the kinetic energy of the eye in motion. Elastic energy in the muscles that control the eye movement will represent the potential energy. In this section we aim at deriving paths taken by the eye if we were to disregard the potential energy; the geodesics of g. They will represent shortest paths between points in ***List*** (equivalently gaze directions) in the context of Riemannian geometry. We will bring in the potential energy and derive controlled paths using optimal control theory later on.

Along the geodesics, tangent vectors are parallel with respect to the Riemannian connection ∇. Let $\sigma(t) = (\theta(t),\phi(t))$ be a geodesic on ***List***. We would have

$$\nabla_{\dot\sigma(t)}\dot\sigma(t) = 0$$

where

$$\dot\sigma(t) = \left(\dot\theta\frac{\partial}{\partial\theta} + \dot\phi\frac{\partial}{\partial\phi}\right)$$

and

$$\nabla_{\dot\sigma(t)}\dot\sigma(t) = \nabla_{\left(\dot\theta\frac{\partial}{\partial\theta}+\dot\phi\frac{\partial}{\partial\phi}\right)}\left(\dot\theta\frac{\partial}{\partial\theta} + \dot\phi\frac{\partial}{\partial\phi}\right).$$

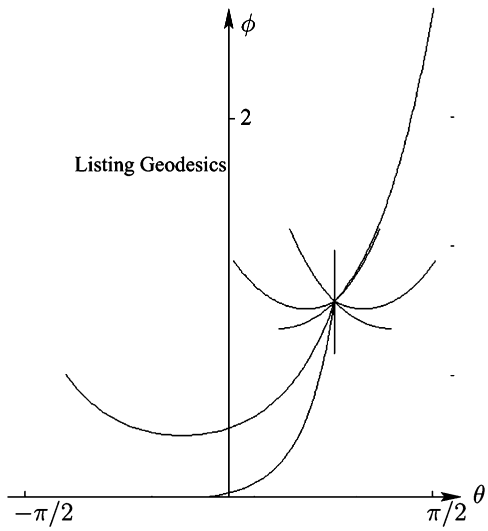

Fig. 2. Geodesics emanating from $(\pi/4, \pi/4)$.

Hence, we would have

$$\ddot{\theta}\frac{\partial}{\partial\theta} + \ddot{\phi}\frac{\partial}{\partial\phi} + \dot{\theta}^2\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} + \dot{\theta}\dot{\phi}\left(\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} + \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta}\right) + \dot{\phi}^2\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = 0.$$

Using

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} &= \sum_{k=1}^{2}\Gamma_{11}^{k}\frac{\partial}{\partial y_k} = -\sin(\phi)\frac{\partial}{\partial\phi}\\
\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} &= \sum_{k=1}^{2}\Gamma_{12}^{k}\frac{\partial}{\partial y_k} = \frac{1}{2\tan(\phi/2)}\frac{\partial}{\partial\theta}\\
\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} &= \sum_{k=1}^{2}\Gamma_{21}^{k}\frac{\partial}{\partial y_k} = \frac{1}{2\tan(\phi/2)}\frac{\partial}{\partial\theta}\\
\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} &= \sum_{k=1}^{2}\Gamma_{22}^{k}\frac{\partial}{\partial y_k} = 0
\end{aligned}$$

where $(y_1, y_2) = (\theta, \phi)$, we obtain the equations of geodesics given as follows:

$$\begin{aligned}
\ddot{\theta} + \frac{1}{\tan(\phi/2)}\dot{\theta}\dot{\phi} &= 0\\
\ddot{\phi} - \sin\phi\dot{\theta}^2 &= 0.
\end{aligned} \tag{2}$$

As an illustrative example, Fig. 2 displays the geodesics emanating from $(\pi/4, \pi/4)$ in the Listing space.

C. Curvature

As we have seen already, the Listing space is a two dimensional manifold, hence an explicit description of its Gauss curvature would provide an intuitive picture of its *shape*. This turns out to be specially true (and somewhat surprising) in our case since the Gauss curvature ends up being a constant. Thus, one may visualize ***List*** locally as a standard two-dimensional sphere.

From the Christoffel symbols we may compute the Riemann curvature tensor $\mathcal{R}$. In terms of the basis[1] $\{\partial_\theta, \partial_\phi\}$, curvature can be written as

$$\mathcal{R}(\partial_\theta, \partial_\phi)\partial_\theta = \nabla_{\partial_\theta}\nabla_{\partial_\phi}\partial_\theta - \nabla_{\partial_\phi}\nabla_{\partial_\theta}\partial_\theta$$

since $[\partial_\theta, \partial_\phi] = 0$, which evaluates to

$$\begin{aligned}
\mathcal{R}(\partial_\theta, \partial_\phi)\partial_\theta &= -\sin^2(\phi/2)\partial_\phi\\
\mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi &= \frac{1}{4}\partial_\theta.
\end{aligned}$$

In particular, the Gauss curvature is given by

$$\begin{aligned}
K(\theta, \phi) &= \frac{\langle \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi\partial_\theta\rangle}{\langle\partial_\theta\partial_\theta\rangle\langle\partial_\phi\partial_\phi\rangle}\\
&= 1
\end{aligned}$$

since

$$[\partial_\phi, \partial_\phi] = \frac{1}{4}.$$

D. General Equations of Motion

Let us write down a potential function in the form $V(\theta, \phi)$ and generalized forces τ_θ, τ_ϕ. Now the Lagrangian is

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2}[g_{11}\dot{\theta}^2 + g_{22}\dot{\phi}^2] - V(\theta, \phi). \tag{3}$$

Hence, from Euler–Lagrange equations, we obtain the equations of motion

$$\begin{aligned}
\ddot{\theta} + \dot{\theta}\dot{\phi}\cot(\phi/2) + \csc^2(\phi/2)\frac{\partial}{\partial\theta}V &= \csc^2(\phi/2)\tau_\theta\\
\ddot{\phi} - \dot{\theta}^2\sin(\phi) + 4\frac{\partial}{\partial\phi}V &= 4\tau_\phi.
\end{aligned} \tag{4}$$

Notice that when $V(\theta, \phi) = 0$, i.e., when the Lagrangian becomes only the kinetic energy, and no forces acting on the system, (4) reduces to that of geodesics given in (2).

VI. Optimal Control

In this section, we discuss the formulation of the eye movement problem as an optimal control problem. We need to incorporate potential energy, muscle forces and damping forces to the model here. There are two sources that contribute to the potential energy; any stored energy due to eye rotation, and energy stored in muscles themselves. The latter has been adequately addressed in existing muscle models. However, there is no discussion in the existing literature on an appropriate formula for the potential energy, and in fact many derivations ignore it completely. In the discussion below, we propose to use two standard muscle models, a simplified linear muscle model as used in [5] and a biomechanically accurate Hill type nonlinear model [26], [27]. Instead of ignoring the stored energy in the eye rotation itself, we fix it quite arbitrarily at $(1/4)\sin^2(\phi/2)$ to represent the fact that frontal gaze direction is the most natural one, and

[1]We use the short notation $\partial_\theta = (\partial)/(\partial\theta)$

as experienced by all of us, rotations by larger angles are increasingly more difficult. Of course we make no claims on the validity of this potential function, and we use it only for illustrative purposes. It is straightforward to replace this particular expression with any physiologically realistic expression when such formulas become available. Another source of arbitrariness in our model is the formulation of the optimal cost function. It is quite standard in biomechanical studies to take the integral of the square of the euclidean norm of muscle activation forces as the cost function (see e.g., [28] and [29]), and we do the same here.

We present three versions of dynamic models with increasing level of complexities. In the first one, we only include potential energy due to eye rotations alone, and set it arbitrarily at $V = (1)/(4)\sin^2(\phi/2)$. The cost function is taken to be the integral of the square of the euclidean norm of generalized torques in the θ and the ϕ directions. The control objective is to control from a given initial gaze direction to a desired final gaze direction in T units of time. Of course this model is highly unrealistic, and only presented for illustrative purposes for the benefit of the readers who may prefer a gentle introduction. The second version adds linear elastic muscles [29], and use a cost function equal to the integral of the square of the euclidean norm of the muscle activation forces. The third model represents muscles using the nonlinear Hill model [26], [27] and uses a cost function equal to the integral of the square of the euclidean norm of the muscle activation forces. In each case, we use the maximum principle to derive necessary conditions in the form of two point boundary value problems, and Matlab routines are used to numerically solve them. We have not explicitly verified that the numerically obtained solutions are indeed optimal. However, noting the fact that the curvature of ***List*** is equal to one, it follows that for initial and final points which are not too far apart, there will be unique extremals joining them, and they are necessarily optimal.

A. Case I: Generalized Torques

For the sake of illustration, we take the potential energy as $V(\theta,\phi) = (1)/(4)\sin^2(\phi/2)$. It follows from (4) that

$$\begin{aligned}\ddot{\theta} + \dot{\theta}\dot{\phi}\cot(\phi/2) &= \csc^2(\phi/2)\tau_\theta\\ \ddot{\phi} - \dot{\theta}^2\sin(\phi) + \frac{1}{2}\sin(\phi) &= 4\tau_\phi.\end{aligned} \tag{5}$$

Let $[z_1, z_2, z_3, z_4]' = [\theta, \dot{\theta}, \phi, \dot{\phi}]'$, then (5) can be written as

$$\frac{d}{dt}\begin{bmatrix} z_1\\ z_2\\ z_3\\ z_4\end{bmatrix} = \begin{bmatrix} z_2\\ -z_2z_4\cot(z_3/2)\\ z_4\\ z_2^2\sin(z_3) - \frac{1}{2}\sin(z_3)\end{bmatrix} + \begin{bmatrix} 0\\ \csc^2(z_3/2)\\ 0\\ 0\end{bmatrix}\tau_\theta + \begin{bmatrix} 0\\ 0\\ 0\\ 4\end{bmatrix}\tau_\phi.$$

Assume that we wish to control the state $(\theta, \dot{\theta}, \phi, \dot{\phi})$ from $(\theta_0, 0, \phi_0, 0)$ to $(\theta_1, 0, \phi_1, 0)$ in T unit of time, while minimizing the control energy

$$\int_0^T [(\tau_\theta(t))^2 + (\tau_\phi(t))^2]\mathrm{d}t.$$

Denoting the costate variable by "λ," let us construct the Hamiltonian as follows:

$$\begin{aligned}\mathcal{H}(z,\lambda) &= \lambda.\dot{z} - \frac{1}{2}\left[\tau_\theta^2 + \tau_\phi^2\right]\\ &= \lambda_1 z_2 - \lambda_2 z_2 z_4\cot(z_3/2) + \lambda_3 z_4\\ &\quad + \lambda_4 z_2^2\sin(z_3) - \frac{1}{2}\lambda_4\sin(z_3)\\ &\quad + \frac{\lambda_2}{\sin^2(z_3/2)}\tau_\theta + 4\lambda_4\tau_\phi\\ &\quad - \frac{1}{2}((\tau_\theta(t))^2 + (\tau_\phi(t))^2).\end{aligned}$$

From Hamilton's equations

$$\dot{z} = \frac{\partial\mathcal{H}}{\partial\lambda}, \quad \dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial z}$$

we obtain

$$\frac{d}{dt}\begin{bmatrix} z_1\\ z_2\\ z_3\\ z_4\\ \lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{bmatrix} = \begin{bmatrix} z_2\\ -z_2z_4\cot(z_3/2) + \csc^2(z_3/2)\tau_\theta^*\\ z_4\\ z_2^2\sin(z_3) - \frac{1}{2}\sin(z_3) + 4\tau_\phi^*\\ 0\\ -\lambda_1 + \lambda_2 z_4\cot(z_3/2) - 2\lambda_4 z_2\sin(z_3)\\ (-\frac{1}{2}\lambda_2 z_2 z_4\csc^2(z_3/2) - \lambda_4 z_2^2\cos z_3 + \\ \frac{1}{2}\lambda_4\cos(z_3) + \lambda_2\cot(z_3/2)\csc^2(z_3/2)\tau_\theta^*)\\ \lambda_2 z_2\cot(z_3/2) - \lambda_3\end{bmatrix}$$

The optimal values of the controls are obtained from the maximum principal as follows:

$$\frac{\partial\mathcal{H}}{\partial\tau_\theta} = 0, \quad \frac{\partial\mathcal{H}}{\partial\tau_\phi} = 0$$

which implies that

$$\begin{aligned}\tau_\theta &= \frac{\lambda_2}{\sin^2(z_3/2)}\\ \tau_\phi &= 4\lambda_4.\end{aligned}$$

Eliminating the control, the state–space system is given by

$$\begin{bmatrix} \dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\\ \dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\\ \dot{\lambda}_4\end{bmatrix} = \begin{bmatrix} z_2\\ -z_2z_4\cot(z_3/2) + \lambda_2\csc^4(z_3/2)\\ z_4\\ z_2^2\sin(z_3) - \frac{1}{2}\sin(z_3) + 16\lambda_4\\ 0\\ -\lambda_1 + \lambda_2 z_4\cot(z_3/2) - 2\lambda_4 z_2\sin(z_3)\\ \Delta\\ \lambda_2 z_2\cot(z_3/2) - \lambda_3\end{bmatrix}$$

where

$$\begin{aligned}\Delta = &-\frac{1}{2}\lambda_2 z_2 z_4\csc^2(z_3/2) - \lambda_4 z_2^2\cos z_3\\ &+\frac{1}{2}\lambda_4\cos(z_3) + \lambda_2^2\cot(z_3/2)\csc^4(z_3/2).\end{aligned}$$

Fig. 3 shows an example of an optimal path in the Listing space and in Fig. 4, Fig. 5 we show the variation of θ, $\dot{\theta}$, ϕ, $\dot{\phi}$ over

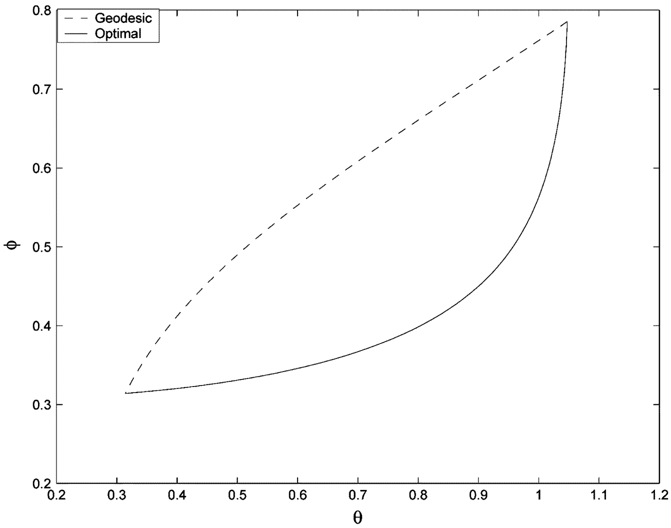

Fig. 3. Optimal path from $(\pi/3, \pi/4)$ to $(\pi/10, \pi/10)$ for Case I.

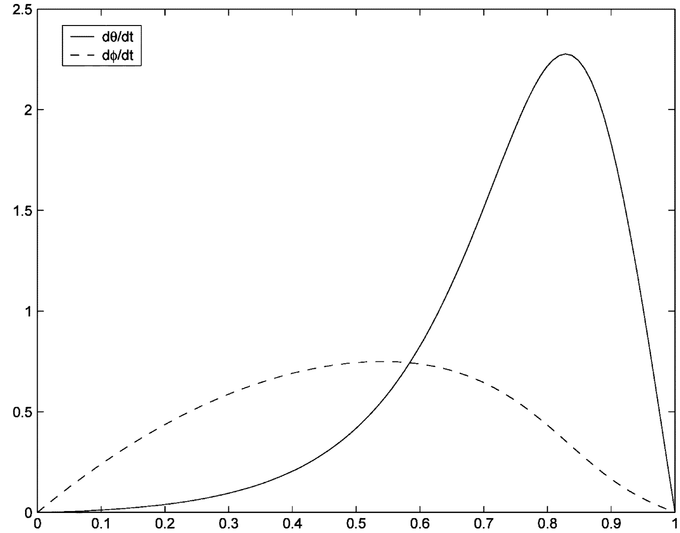

Fig. 5. $\dot{\theta}$ and $\dot{\phi}$ as a function of time for the path in Fig. 3.

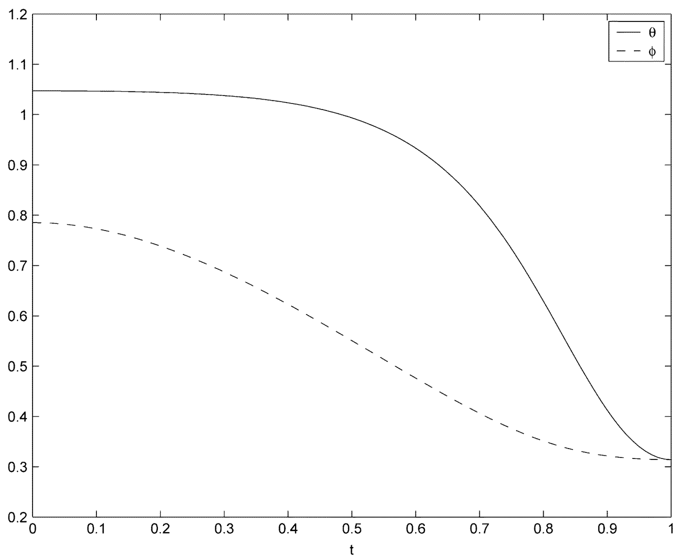

Fig. 4. θ and ϕ as a function of time for the path in Fig. 3.

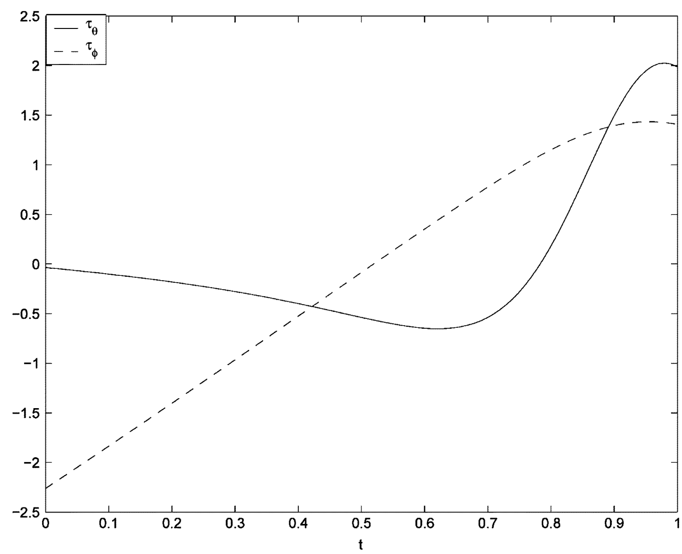

Fig. 6. $4\tau_\theta, 4\tau_\phi$ as a function of time for the path in Fig. 3.

time. Finally, in Fig. 6, we show the corresponding torques τ_θ and τ_ϕ.

B. Case II: Simplified Muscles

In this subsection, we consider a linearized model for each of the three pairs of musculotendons that are actuating the eye rotation. We would assume that each of the musculotendons consists of a linear spring with spring constant k_i, a damper with damping constant b_i, and an active force F_i, where $i = 1 \ldots 6$. Our first task is to derive the generalized torques as a function of the forces acting on the musculotendons. Let us describe the changes in θ and ϕ by $\theta \longrightarrow \theta + \delta\theta$ and $\phi \longrightarrow \phi$.

The total virtual work performed by the spring, damper, and the active forces would be given by

$$\sum_{i=1}^{6}[F_i + C_i]\frac{\partial l_i}{\partial \theta}\, d\theta$$

where

$$C_i = k_i\left(l_i - l_{i_0}\right) + b_i\left(\dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi}\right). \tag{6}$$

It would follow that the generalized torques τ_θ (and likewise τ_ϕ) are given by

$$\tau_\theta = \sum_{i=1}^{6}[F_i + C_i]\frac{\partial l_i}{\partial \theta}$$
$$\tau_\phi = \sum_{i=1}^{6}[F_i + C_i]\frac{\partial l_i}{\partial \phi}. \tag{7}$$

The optimal control problem can be posed by requiring to minimize

$$\frac{1}{2}\int_0^T \sum_{i=1}^{6} F_i^2 \mathrm{d}t. \tag{8}$$

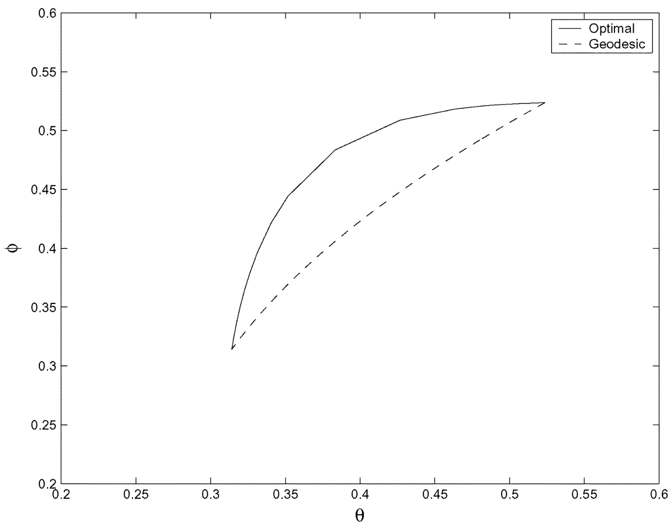

Fig. 7. Optimal path and the geodesic in the (θ, ϕ) space from $(\pi/6, \pi/6)$ to $(\pi/10, \pi/10)$ for Case II.

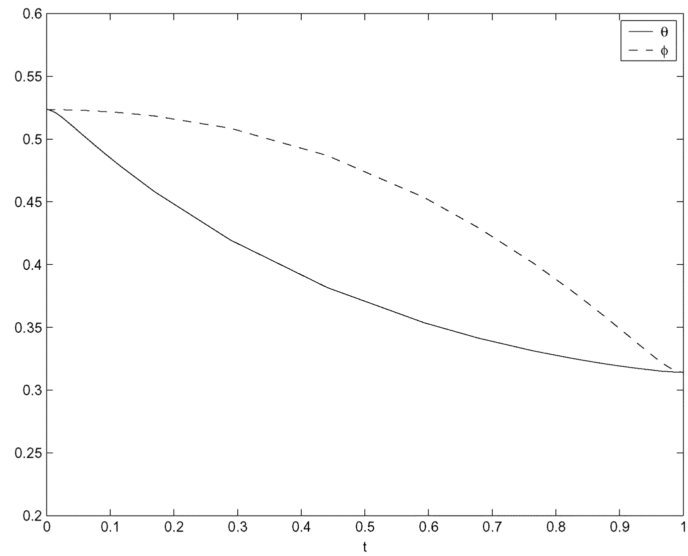

Fig. 8. θ and ϕ as a function of time for the path in Fig. 7.

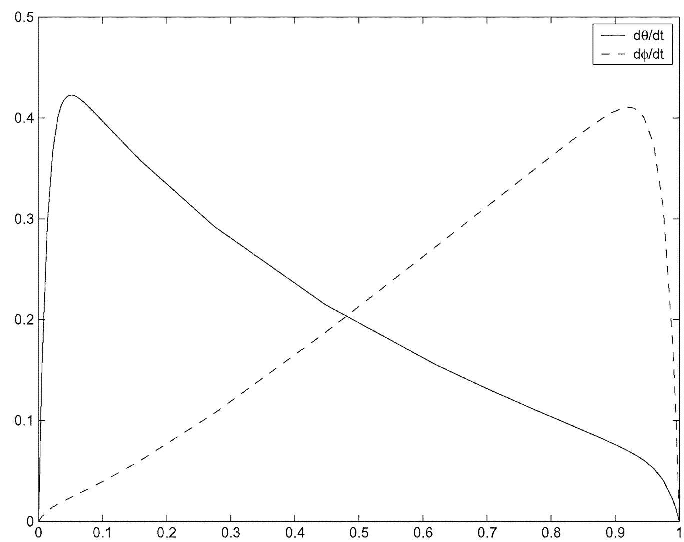

Fig. 9. $\dot{\theta}$ and $\dot{\phi}$ as a function of time for the path in Fig. 7.

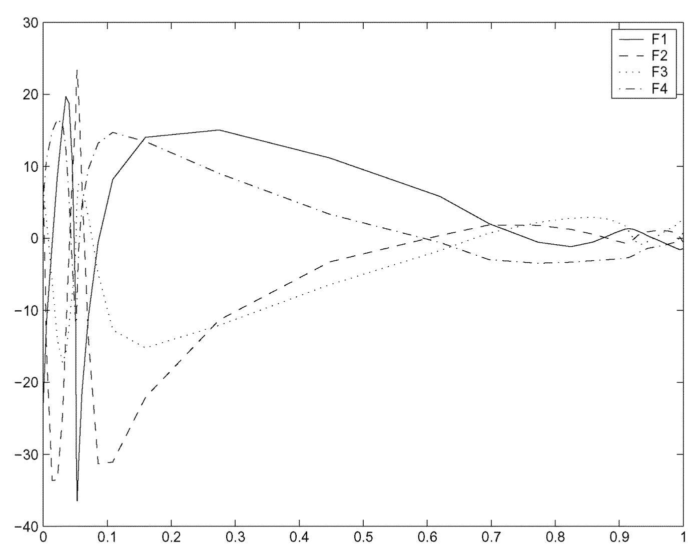

Fig. 10. Four Rectus Muscle force variations for the path in Fig. 7. The oblique muscles have not been shown in this figure.

We construct the Hamiltonian given by

$$\begin{aligned}\mathcal{H}(z,\lambda) &= \lambda_1 z_2 - \lambda_2 z_2 z_4 \cot(z_3/2) + \lambda_3 z_4 \\ &\quad + \lambda_4 z_2^2 \sin(z_3) - \frac{1}{2}\lambda_4 \sin(z_3) \\ &\quad + \frac{\lambda_2}{\sin(z_3/2)} \sum_{i=1}^{6} [F_i + C_i] \frac{\partial l_i}{\partial \theta} \\ &\quad + 4\lambda_4 \sum_{i=1}^{6} [F_i + C_i] \frac{\partial l_i}{\partial \phi} - \frac{1}{2}\int_0^T \sum_{i=1}^{6} F_i^2 \mathrm{d}t.\end{aligned}$$

From the 'Maximum Principle' we obtain the following:

$$F_i^* = \lambda_2 \csc(z_3/2) \frac{\partial l_i}{\partial \theta} + 4\lambda_4 \frac{\partial l_i}{\partial \phi}. \tag{9}$$

The generalized torques τ_θ^* and τ_ϕ^* can be computed from (7) and (9). We now proceed to find $(\partial l_i)/(\partial \theta)$ and $(\partial l_i)/(\partial \phi)$ as follows. Let q_i and $p_i(t)$ be the fixed end and the point of attachment to the eye, of the muscle. Then, we have

$$l_i^2(t) = (p_i(t) - q_i)^T (p_i(t) - q_i)$$

where $p_i(t) = \boldsymbol{R} p_i(0)$ and $\boldsymbol{R}$ is the 3×3 rotation matrix. We obtain

$$\begin{aligned} l_i^2(t) &= (\boldsymbol{R} p_i(0) - q_i)^T (\boldsymbol{R} p_i(0) - q_i) \\ &= p_i^T(0) p_i(0) + q_i q_i^T - 2 p_i^T(0) \boldsymbol{R}^T q_i. \end{aligned} \tag{10}$$

Therefore

$$\begin{aligned} \frac{\partial l_i}{\partial \theta} &= -p_i^T(0) \left(\frac{\partial \boldsymbol{R}^T}{\partial \theta} \right) q_i / l_i \\ \frac{\partial l_i}{\partial \phi} &= -p_i^T(0) \left(\frac{\partial \boldsymbol{R}^T}{\partial \phi} \right) q_i / l_i. \end{aligned} \tag{11}$$

One can easily write down the system with state z and costate λ similar to the one obtained in (VI.A). In Figs. 7–10 the optimal path and the corresponding rectus muscle forces have been shown.

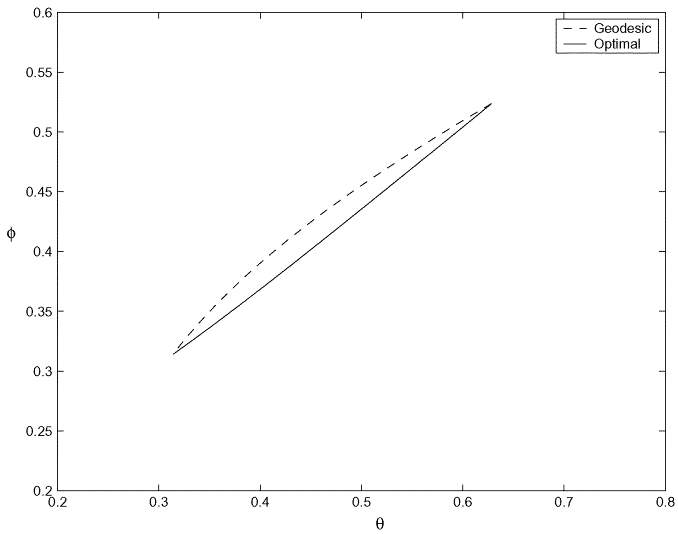

Fig. 11. Optimal path and the geodesic in the (θ, ϕ) space from $(\pi/5, \pi/6)$ to $(\pi/10, \pi/10)$ for Case III.

C. Case III: Hill-Type Muscles

Hill-type muscle models have been described in detail by Zajak in [26]. Formulation of the optimal control problem follows very closely to that given in Section VI-B. One obtains the generalized torques τ_θ and τ_ϕ as follows:

$$\begin{aligned} \tau_\theta &= \sum_{i=1}^{6} F^i_{\mathrm{total}} \frac{\partial l_i}{\partial \theta} \\ \tau_\phi &= \sum_{i=1}^{6} F^i_{\mathrm{total}} \frac{\partial l_i}{\partial \phi} \end{aligned} \tag{12}$$

where

$$F^i_{\mathrm{total}} = F^i_t - (F^i_{\mathrm{act}} + F^i_{pe} + B^i_m \dot{l}_i).$$

The terms $F_t, F_{\mathrm{act}}, F_{pe}, B_m$ above are described in [29] and the superscript i is the index for each muscle.

We would minimize the active force in the muscle $F^i_{\mathrm{act}}(t)$ by minimizing

$$\int_0^T \sum_{i=1}^{6} \left[F^i_{\mathrm{act}}(t)\right]^2 \mathrm{d}t.$$

Fig. 11 shows the geodesic and the optimal path on the Listing's space. Parameters for oblique muscles were chosen such that they have a very small activity and are not shown. In Figs. 12–15, we show only the corresponding rectus muscle activities.

As an example, in Table I we present, numerical results to compare lengths of minimal eye rotations with and without the Listing's constraint. In the case when the Listing's law is observed, we compute the geodesic distances as well as distances along curves that minimize the energy function considered in the Section VI.A (see Table I). Lengths of eye rotation using the Hill type muscles have not been shown here.

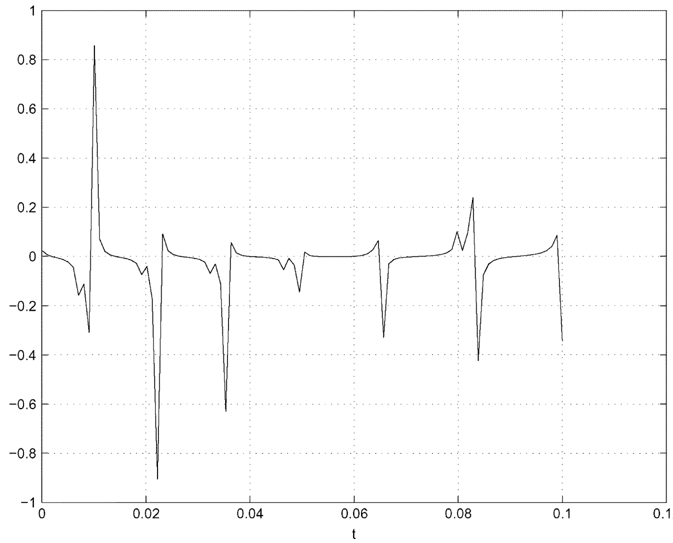

Fig. 12. Lateral rectus muscle activity for the path in Fig. 11.

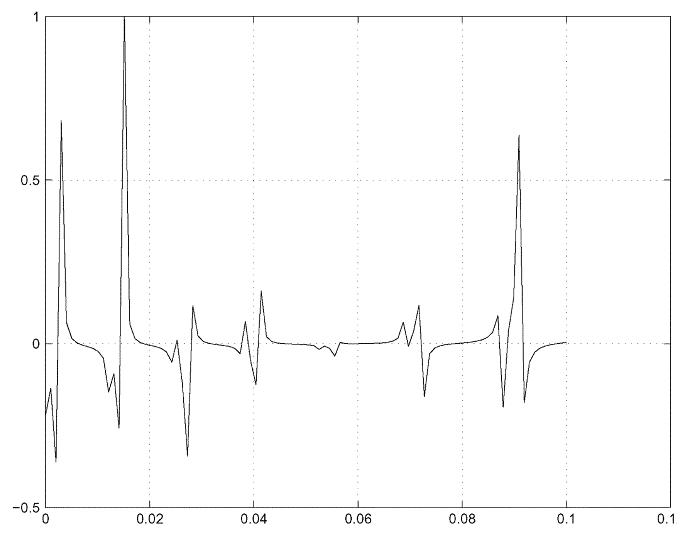

Fig. 13. Medial rectus muscle activity for the path in Fig. 11.

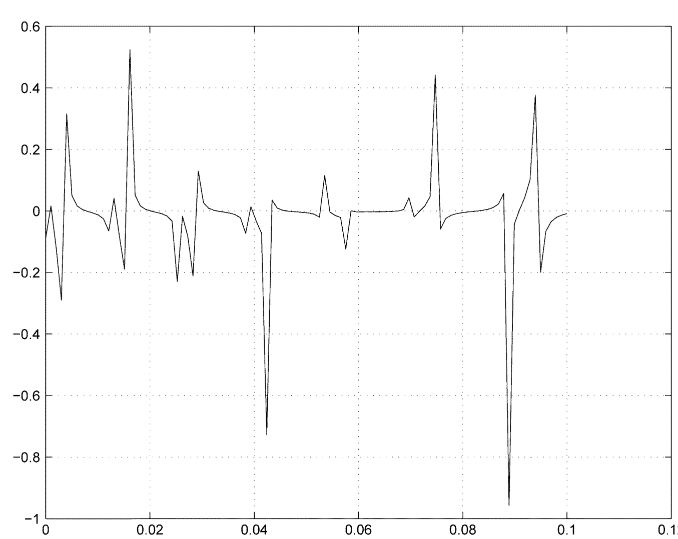

Fig. 14. Superior rectus muscle activity for the path in Fig. 11.

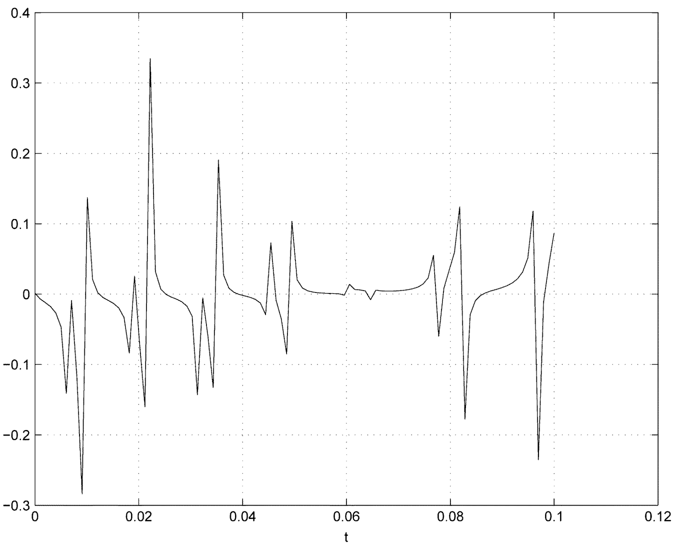

Fig. 15. Inferior rectus muscle activity for the path in Fig. 11.

TABLE I
COMPARISON OF LENGTHS OF EYE ROTATIONS

From (θ, ϕ)	**To** (θ, ϕ)	**distance** (radians) $SO(3)$	**Geodesic on List**	**Min. energy on List**
$(\frac{\pi}{4}, \frac{\pi}{6})$	$(\frac{\pi}{8}, \frac{\pi}{8})$	0.219	0.222	0.324
$(\frac{\pi}{4}, \frac{\pi}{4})$	$(\frac{\pi}{8}, \frac{\pi}{6})$	0.359	0.368	0.368
$(\frac{\pi}{6}, \frac{\pi}{10})$	$(\frac{\pi}{8}, \frac{\pi}{4})$	0.476	0.480	0.482

VII. CONCLUDING REMARKS

In this paper, a relatively complete description of the Riemannian geometry of the space of eye rotations subject to the Listing's law is given. It is shown that the configuration space is diffeomorphic to the real projective space $\mathbb{RP}^2$ and has constant positive curvature equal to 1. The system is presented as a simple mechanical control system subjected to a holonomic constraint (provided by Listing's law). Table I, is given merely to get an idea of the trajectory that eye would choose when directing the gaze from one direction to another. It is not the shortest path obtained on $\mathsf{SO}(3)$, but a different path under the Listing constraint.

Different models of musculotendons, as well as how to compute the torques derived from the forces in these musculotendons on to the configuration manifold ***List***, are also discussed in Section VI. In particular, few examples with different musculotendon complexes are given merely to propose possible control strategies of the motor cortex. It remains to be validated by actual eye movement and neural recordings which control strategy the brain really undertakes. We found that such experimental data is hardly available as yet. If saccadic eye movement (the rapid eye movements which are the fastest movements of an external part of the human body with peak angular speed reaching up to 1000 degrees per second) is the topic of interest, the likely strategy would be to minimize the time instead of energy.

REFERENCES

[1] D. Robinson, "The mechanics of human saccadic eye movement," *J. Physiol.*, vol. 174, pp. 245–264, 1964.
[2] J. B. Nielsen, "How we walk: Central control of muscle activity during human walking," *Neurosci.*, vol. 9, no. 3, pp. 195–204, Jun. 2003.
[3] C. Martin and L. Schovanec, "Muscle mechanics and dynamics of ocular motion," *J. Math. Syst., Estimat. Control*, vol. 8, pp. 1–15, 1998.
[4] J. Miller and D. Robinson, "A model of the mechanics of binocular alignment," *Comput. Biomed. Res.*, vol. 17, pp. 436–470, 1984.
[5] T. Raphan, "Modeling control of eye orientation in three dimensions. 1. role of muscle pulleys in determining saccadic trajectory," *J. Neurophysiol.*, vol. 79, pp. 2653–2667, 1998.
[6] C. Quaia and L. Optican, "Commutative saccadic generator is sufficient to control a 3-d ocular plant with pulleys," *J. Neurophysiol.*, vol. 79, pp. 3197–3215, 1998.
[7] R. M. Murray, "Nonlinear control of mechanical systems: A Lagrangian perspective," *Annu. Rev. Control*, vol. 21, pp. 31–45, 1997.
[8] H. Goldstein, *Classical Mechanics*. Reading, MA: Addison-Wesley, 1980.
[9] S. Smale, "Topology and mechanics, I.," *Inventiones Mathematicae*, vol. 10, pp. 305–331, 1970.
[10] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd ed. Reading, MA: Addison Wesley, 1987.
[11] P. S. Krishnaprasad, R. Yang, and W. P. Dayawansa, "Control problems on principal bundles and nonholonomic mechanics," in *Proc. 30th IEEE Conf. Decision Control*, Dec. 1991, vol. 2, pp. 1133–1138.
[12] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*. New York: Springer-Verlag, 1999.
[13] V. I. Arnol'd, *Mathematical Methods of Classical Mechanics*, ser. Graduate Texts in Mathematics, 2nd ed. New York: Springer-Verlag, 1989, 60.
[14] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*. New York: Springer-Verlag, 2004.
[15] R. W. Brocket, "Feedback invariants for nonlinear systems," in *Proc. 6th IFAC Congr.*, Helsinki, Finland, 1978, pp. 1115–1120.
[16] A. Isidori, *Nonlinear Control Systems*, 3rd ed. New York: Springer-Verlag, 1997.
[17] H. J. Sussmann, "A general theorem on local controllability," *SIAM J. Control Optim.*, vol. 25, no. 2, pp. 158–194, 1987.
[18] D. Tweed and T. Villis, "Geometric relations of eye position and velocity vectors during saccades," *Vision Res.*, vol. 30, pp. 111–127, 1990.
[19] A. A. Handzel and T. Flash, "The geometry of eye rotations and Listing's law," in *Advances in Neural Information Processing Systems*, D. S. Touretzky, M. C. Mozer, and M. E. Hasselmo, Eds. Cambridge, MA: MIT Press, 1996, vol. 8, pp. 117–123.
[20] T. Haslwanter, "Mathematics of three-dimensional eye rotations," *Vision Res.*, vol. 35, pp. 1727–1739, 1995.
[21] K. Hepp, "On Listing's law," *Commun. Math. Phys.*, vol. 132, pp. 285–292, 1990.
[22] J. V. Opstal, "Three-dimensional kinematics underlying gaze control," in *Models of Neural Networks TV*, E. Domany, J. L. V. Hemman, and K. Schultan, Eds. New York: Springer Verlag, Feb. 1988.
[23] J. Kuipers, *Quaternions and Rotation Sequences*. Princeton, NJ: Princeton Univ. Press, 1998.
[24] W. M. Boothby, *An Introduction to Differentiate Manifolds and Riemannian Geometry*. San Diego, CA: Academic Press, 1986.
[25] M. P. do Carmo, *Riemannian Geometry*. Boston, MA: Birkhäuser, 1993.
[26] F. Zajac, "Muscle and tendon: Properties, models, scaling, and application to biomechanics and motor control," *CRC Critical Rev. Biomed. Eng.*, vol. 17, pp. 359–511, 1989.
[27] A. Hill, "The heat of shortening and dynamic constants of muscle," in *Proc. Roy. Soc. B.*, 1938, vol. 126, pp. 136–195.
[28] D. Tweed, T. Haslwanter, and M. Fetter, "Optimizing gaze control in three dimensions," *Science*, vol. 281, pp. 1363–1366, 1998.
[29] C. Martin and L. Schovanec, "The control and mechanics of human movement systems," *Prog. Syst. Control Theory*, vol. 25, pp. 173–202, 1999.

Ashoka D. Polpitiya (S'03–M'04) received the B.Sc. degree in electrical and electronic engineering from the University of Peradeniya, Sri Lanka, in 1996, and the M.S. and D.Sc. degrees in systems science and mathematics from Washington University, St. Louis, MO, in 2000 and 2004, respectively.

From 2004 to 2006, he was a Postdoctoral Research Associate with the Cellular Injury and Adaptation Laboratory at Washington University, School of Medicine. Currently, he is a Senior Research Scientist at the Pacifc Northwest National Laboratory, Richland, WA, operated by Battelle for the US Department of Energy. His research interests are in the areas of systems biology, application of control theory in biological systems, and geometric control.

Wijesuriya P. Dayawansa (F'06) received the B.Sc. degree in electrical engineering from University of Peradeniya, Peradeniya, Sri Lanka, in 1978, the M.Sc. degree in electrical engineering from Clarkson University, Potsdam, NY, in 1982, and the D.Sc. degree in systems science and mathematics from Washington University, St. Louis, MO, in 1986.

He is currently a Professor of Mathematics at Texas Tech University, Lubbock. His research interests include the areas of mathematical theory of nonlinear control systems and learning and intelligent systems.

Clyde F. Martin (M'74–SM'83–F'91) received the B.S. degree from Kansas State Teachers College, Emporia, in 1965, and the M.Sc. and Ph.D. degrees from the University of Wyoming, Laramie, in 1967 and 1971, respectively.

He has held regular academic positions at Utah State University, Case Western Reserve University, and Texas Tech University, where he is currently the Paul Whitfield Horn Professor and the Ex-Students Distinguished Professor of Mathematics. He is also the Director of the Institute for the Mathematics of Life Sciences at Texas Tech University. He has held postdoctoral or visiting positions at Harvard University and the Royal Institute of Technology in Stockholm, Sweden, and has held two postdoctoral positions at NASA Ames Research Center. His primary technical interests include the application of mathematical and control theoretical techniques to problems in the life sciences. He has published numerous papers, monographs, and edited conference proceedings.

Dr. Martin is currently Vice Chair of the SIAM Activity Group in Control Theory and is Editor-in-Chief of the *Journal of Mathematical Systems, Estimation, and Control*.

Bijoy K. Ghosh (S'79–M'83–SM'90–F'00) received the B.Tech. and M.Tech. degrees in electrical and electronics engineering from BITS, Pilani, the Indian Institute of Technology, Kanpur, India, and the Ph.D. degree in engineering from the Decision and Control Group of the Division of Applied Sciences, Harvard University, Cambridge, MA, in 1977, 1979, and 1983, respectively.

From 1983 to 2006, he was with the Department of Electrical and Systems Engineering, Washington University, St. Louis, MO, where he was a Professor and Director of the Center for BioCybernetics and Intelligent Systems. Currently, he is the Brooks Regents Professor of Mathematics and Statistics at Texas Tech University, Lubbock. His research interests are in multivariable control theory, machine vision, robotic manufacturing, and biosystems and control.