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# Parameterizing the Eye Movement Dynamics on LIST and SO(3): How to move the singularities away

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**Abstract:** Using a suitable parametrization, the dynamics of eye rotation can be written as an Euler Lagrange equation on LIST or SO(3) depending on whether eye is assumed to satisfy the Listing's constraint or not. The equations typically have singularities and we show in this paper that by choosing alternate forms of the state variables, the singularities can be moved, for example from the frontal gaze to the backward gaze direction in the case of LIST. Similar constructions are also possible on SO(3).

**Key Words:** Eye Movement, Listing's Law, Euler Lagrange Equation, singularities.

## 1 Introduction

Modeling the eye plant, in order to generate various eye movements, has been one of the important goals among neurologists, physiologists and engineers for a long time. Since as early as 1845 (e.g. work of Listing, Donders, Helmholtz etc.), models have been written to understand various eye movements [1], [2], [3]. Earlier studies which used modeling as a means of understanding the control of the eye movements have adopted two main approaches. One focusing on the details of the properties of the extraocular muscles [4], [5], and the other focusing on control mechanisms using over simplified linear models [6]. In spite of several notable studies of the three dimensional eye movements [7], [8], [9], there has not been a rigorous treatment of the topic in the framework of modern control theory and geometric mechanics [10], [11], [12]; until the recent work in [13], [14].

Eye movement dynamics can be modeled as a mechanical system by treating the eye as a homogeneous sphere rotating about its center of mass. It has been shown in [13], [14] and [15] that in this case the movement dynamics is written by extremizing a Lagrangian, giving rise to what is known as an Euler Lagrange's Equation. In general such an equation has singularities and the content of this paper is to show that one can move the locations of these singularities by suitably choosing alternate forms of the state variables. On SO(3), for example, four charts are chosen that cover the space. Choosing a suitable set of state variables, eye movement dynamics have been described on each of these charts and the resulting Euler Lagrange's equations are shown not to have a singularity on each of the charts. On LIST, a submanifold of SO(3) (see [14]), the same construction is possible using only two charts. A suitable optimal control problem is considered and the corresponding costate equations are derived in this paper, for LIST and SO(3). We conclude by remarking that clearly it is desirable to locate the singularities in a space restricted by the physiological limits of the eye muscles, viz. the gaze pointing towards the back side of the eye or eye rotating under pure torsion. Using the axis angle parametrization [16], the paper introduces a good chart, one for LIST and one for SO(3), for which the singularities are appropriately located.

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## 2 Riemannian metric on LIST and SO(3)

The space SO(3) of all orientations can be parameterized using angle variables $\theta$, $\phi$, $\alpha$ as a $3 \times 3$ special orthogonal matrix given by (1) (see top of next page), which can be obtained as a surjective map from $S^3$, the space of unit quaternions (see [14] for details). Unit quaternions can be parameterized as

$$q = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\theta\,\cos\alpha & \sin\frac{\phi}{2}\sin\theta\,\cos\alpha & \sin\frac{\phi}{2}\sin\alpha \end{pmatrix}, \tag{2}$$

where $\phi$ measures the counterclockwise rotation about the unit vector (axis)

$$n = \begin{pmatrix} \cos\theta\,\cos\alpha & \sin\theta\,\cos\alpha & \sin\alpha \end{pmatrix}. \tag{3}$$

It is often assumed that the eye orientation matrices satisfy Listing's constraint, which is described by $\alpha = 0$. The corresponding rotation matrices and unit quaternions easily follows from (1) and (2). The Riemannian metric on SO(3)[1] is given by

$$g = \sin^2\left(\frac{\phi}{2}\right)\cos^2(\alpha)d\theta^2 + \sin^2\left(\frac{\phi}{2}\right)d\alpha^2 + \frac{1}{4}d\phi^2. \tag{4}$$

The Riemannian metric on LIST, where $\alpha = 0$, is obtained analogously by

$$g = \sin^2\left(\frac{\phi}{2}\right)d\theta^2 + \frac{1}{4}d\phi^2. \tag{5}$$

The geodesic equation on SO(3)[2] is given by

$$\begin{aligned} \ddot\theta + \dot\theta\dot\phi\cot\left(\frac{\phi}{2}\right) - 2\dot\theta\dot\alpha\tan(\alpha) &= 0, \\ \ddot\phi - (\dot\theta)^2\sin(\phi)\cos^2(\alpha) - (\dot\alpha)^2\sin(\phi) &= 0, \\ \ddot\alpha + \frac{(\dot\theta)^2}{2}\sin(2\alpha) + \dot\phi\dot\alpha\cot\left(\frac{\phi}{2}\right) &= 0. \end{aligned} \tag{6}$$

Likewise, the geodesic equation on LIST is given by

$$\ddot\theta + \dot\theta\dot\phi\cot\left(\frac{\phi}{2}\right) = 0, \quad \ddot\phi - (\dot\theta)^2\sin(\phi) = 0. \tag{7}$$

[1] Details of this calculation has been sketched in [13].
[2] Requires calculating the associated Christoffel symbols.

$$\begin{pmatrix} \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\,(1 - 2\cos^2\alpha\,\cos^2\theta) & \sin^2\frac{\phi}{2}\,\cos^2\alpha\,\sin(2\theta) - \sin\phi\,\sin\alpha & \sin^2\frac{\phi}{2}\,\sin(2\alpha)\,\cos\theta + \sin\phi\,\cos\alpha\,\sin\theta \\ \sin^2\frac{\phi}{2}\,\cos^2\alpha\,\sin(2\theta) + \sin\phi\,\sin\alpha & \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\,(1 - 2\cos^2\alpha\,\sin^2\theta) & \sin^2\frac{\phi}{2}\,\sin(2\alpha)\,\sin\theta - \sin\phi\,\cos\alpha\,\cos\theta \\ \sin^2\frac{\phi}{2}\,\sin(2\alpha)\,\cos\theta - \sin\phi\,\cos\alpha\,\sin\theta & \sin^2\frac{\phi}{2}\,\sin(2\alpha)\,\sin\theta + \sin\phi\,\cos\alpha\,\cos\theta & \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\,\cos(2\,\alpha) \end{pmatrix} \quad (1)$$

Note that the above two geodesic equations (6), (7) are singular when $\sin(\frac{\phi}{2}) = 0$. Additionally, (6) is singular when $\cos(\alpha) = 0$. When $\phi$ is a multiple of $2\pi$, i.e. when the gaze is pointing straight to the front, we have a singularity. Likewise, when $\alpha$ is an odd multiple of $\frac{\pi}{2}$, i.e. when the axis of rotation is perpendicular to the Listing's plane, we have a singularity.

We propose to move the singularity associated with the geodesic equation (7), to a location that is beyond the physiological limits of attainable eye positions, by appropriately defining state variables. For example, if we define

$$[z_1, z_2, z_3, z_4] = \left[\theta, \dot{\theta}, \phi, \dot{\phi}\right] \quad (8)$$

and assume that $\sin\left(\frac{\phi}{2}\right) \neq 0$, the geodesic equation (7) can be written as

$$\begin{aligned} \dot{z}_1 &= z_2, \;\; \dot{z}_2 = -\, z_2 z_4 \cot\left(\frac{z_3}{2}\right), \\ \dot{z}_3 &= z_4, \;\; \dot{z}_4 = z_2^2 \sin(z_3). \end{aligned} \quad (9)$$

On the other hand, if we define

$$[\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4] = \left[\theta, \frac{\dot{\theta}}{\tan\frac{\phi}{2}}, \phi, \frac{\dot{\phi}}{\tan(\frac{\phi}{2})}\right] \quad (10)$$

and assume that $\cos\left(\frac{\phi}{2}\right) \neq 0$, the geodesic equation (7) can be written as

$$\begin{aligned} \dot{\bar{z}}_1 &= \bar{z}_2 \tan\left(\frac{\bar{z}_3}{2}\right), \;\; \dot{\bar{z}}_2 = -\bar{z}_2\bar{z}_4\left(1 + \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right), \\ \dot{\bar{z}}_3 &= \bar{z}_4 \tan\left(\frac{\bar{z}_3}{2}\right), \;\; \dot{\bar{z}}_4 = \bar{z}_2^2 \sin(\bar{z}_3)\tan\left(\frac{\bar{z}_3}{2}\right) - \frac{\bar{z}_4^2}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right) \end{aligned} \quad (11)$$

Note that $\sin\left(\frac{\phi}{2}\right) = 0$ corresponds to a gaze pointing straight to the front, and $\cos\left(\frac{\phi}{2}\right) = 0$ corresponds to a gaze pointing straight to the back. Thus, transforming from (9) to (11), the singularity has been moved from the front to the back. A similar construction is possible for the geodesic equation (6) on SO(3).

## 3 Dynamical equation of motion on LIST

The Riemannian metric (5), on LIST enables us to write down an expression for the kinetic energy. In general, the dynamics is affected by an additional potential energy and an external input torque. We assume that the potential energy term is absent and the axis of rotation is restricted to the Listing's plane i.e. $\alpha = 0$. We denote the generalized torques as $\tau_\theta$ and $\tau_\phi$ and express the Lagrangian as

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2}\left[\sin^2\left(\frac{\phi}{2}\right)\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2\right].$$

The dynamical equation of motion is described by the Euler Lagrange equations,

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\beta}}\right) - \left(\frac{dL}{d\beta}\right) = \tau_\beta, \quad (12)$$

where $\beta$ is the angle variable. On LIST, the dynamical equation of motion is given by

$$\begin{aligned} \ddot{\theta} + \dot{\theta}\dot{\phi}\cot\left(\tfrac{\phi}{2}\right) &= \csc^2\left(\tfrac{\phi}{2}\right)\tau_\theta, \\ \ddot{\phi} - \dot{\theta}^2\,\sin(\phi) &= 4\tau_\phi. \end{aligned} \quad (13)$$

Defining the state variables as in (8), and assuming that $\sin(\frac{\phi}{2}) \neq 0$, the dynamical equation of motion (13) can be written as

$$\begin{aligned} \dot{z}_1 &= z_2, \;\; \dot{z}_2 = -\, z_2 z_4 \cot\left(\frac{z_3}{2}\right) + \csc^2\left(\frac{z_3}{2}\right)\tau_\theta, \\ \dot{z}_3 &= z_4, \;\; \dot{z}_4 = z_2^2\,\sin(z_3) + 4\tau_\phi. \end{aligned} \quad (14)$$

On the other hand, by defining the state variables as in (10), and assuming that $\cos(\frac{\phi}{2}) \neq 0$, the same equation of motion (13) is written as

$$\begin{aligned} \dot{\bar{z}}_1 &= \bar{z}_2 \tan\left(\frac{\bar{z}_3}{2}\right), \;\; \dot{\bar{z}}_3 = \bar{z}_4 \tan\left(\frac{\bar{z}_3}{2}\right), \\ \dot{\bar{z}}_2 &= -\bar{z}_2\bar{z}_4\left(1 + \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right) + \bar{\tau}_\theta \sec^2\left(\frac{\bar{z}_3}{2}\right), \\ \dot{\bar{z}}_4 &= \bar{z}_2^2\sin(\bar{z}_3)\tan\left(\frac{\bar{z}_3}{2}\right) - \frac{\bar{z}_4^2}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right) + 4\bar{\tau}_\phi, \end{aligned} \quad (15)$$

where the generalized torques $\bar{\tau}_\theta$ and $\bar{\tau}_\phi$ are defined as

$$\bar{\tau}_\theta = \cot^3\left(\frac{\phi}{2}\right)\tau_\theta, \;\; \bar{\tau}_\phi = \cot\left(\frac{\phi}{2}\right)\tau_\phi.$$

Note that the generalized torques had to be redefined, for otherwise (15) would still remain singular on the open interval $\cos(\frac{\phi}{2}) \neq 0$. The problem of redefining the generalized torque is that on the intersection of the two charts $\sin(\frac{\phi}{2}) \neq 0$ and $\cos(\frac{\phi}{2}) \neq 0$, the generalized torques $\tau_\theta$, $\bar{\tau}_\theta$ and $\tau_\phi$, $\bar{\tau}_\phi$ do not match. Thus when the states move from one chart to another, the external torque inputs have to be switched. In order to circumvent the problem of switching the external input between charts, we define a new variable as follows:

$$\begin{aligned} \eta_\theta &= \frac{\tau_\theta}{\sin^3\left(\frac{\phi}{2}\right)} = \frac{\bar{\tau}_\theta}{\cos^3\left(\frac{\phi}{2}\right)}, \\ \eta_\phi &= \frac{\tau_\phi}{\sin\left(\frac{\phi}{2}\right)} = \frac{\bar{\tau}_\phi}{\cos\left(\frac{\phi}{2}\right)}. \end{aligned}$$

The equations of motion (13) can now be written when $\sin(\frac{\phi}{2}) \neq 0$ as

$$\begin{aligned} \dot{z}_1 &= z_2, \;\; \dot{z}_2 = -z_2 z_4 \cot\left(\frac{z_3}{2}\right) + \sin\left(\frac{z_3}{2}\right)\eta_\theta, \\ \dot{z}_3 &= z_4, \;\; \dot{z}_4 = z_2^2\sin(z_3) + 4\sin\left(\frac{z_3}{2}\right)\eta_\phi. \end{aligned} \quad (16)$$

Likewise, assuming that $\cos(\frac{\phi}{2}) \neq 0$, the same equation of motion (13) is written as

$$\begin{aligned} \dot{\bar{z}}_1 &= \bar{z}_2 \tan\left(\frac{\bar{z}_3}{2}\right), \;\; \dot{\bar{z}}_3 = \bar{z}_4 \tan\left(\frac{\bar{z}_3}{2}\right), \\ \dot{\bar{z}}_2 &= -\bar{z}_2\bar{z}_4\left(1 + \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right) + \eta_\theta \cos\left(\frac{\bar{z}_3}{2}\right), \\ \dot{\bar{z}}_4 &= \bar{z}_2^2\sin(\bar{z}_3)\tan\left(\frac{\bar{z}_3}{2}\right) - \frac{\bar{z}_4^2}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right) + 4\eta_\phi \cos\left(\frac{\bar{z}_3}{2}\right). \end{aligned} \quad (17)$$

The important point is that during chart switching, the dynamical system switches from (16) to (17). However the control variables $\eta_\theta$ and $\eta_\phi$ do not switch.

## 4 Optimal control of eye movement under the Listing's constraint

In the previous section, we have defined a dynamical system (7) using variables $(\theta, \dot{\theta}, \phi, \dot{\phi})$. In two different charts, we show in (16) and (17) that the system (7) can be equivalently written without any singularities using control variables $\eta_\theta$ and $\eta_\phi$. A standard optimal control problem would be to steer the state $(\theta, \dot{\theta}, \phi, \dot{\phi})$ from $(\theta_0, 0, \phi_0, 0)$ to $(\theta_1, 0, \phi_1, 0)$ in $T$ units of time while minimizing the control energy

$$\frac{1}{2}\int_0^T \left(\eta_\theta^2 \; + \; \eta_\phi^2\right) dt.$$

One way to handle the optimal control problem is to define state vector $z = (z_1, z_2, z_3, z_4)$ and costate variable vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and define a Hamiltonian function $H(z, \lambda)$ as follows:

$$H(z, \lambda) \; = \; \lambda \dot{z}^T \; - \; \frac{1}{2}(\eta_\theta^2 \; + \; \eta_\phi^2). \tag{18}$$

We can now write down the Hamiltonian equation as follows:

$$\dot{z} \; = \; \frac{\partial H}{\partial \lambda}, \;\; \dot{\lambda} \; = \; -\frac{\partial H}{\partial z}. \tag{19}$$

The $\dot{z}$ equation is simply the state equation (16). The $\dot{\lambda}$ equation describes the costate dynamics. The optimal control is computed by setting

$$\frac{\partial H}{\partial \eta_\theta} \; = \; 0; \;\; \frac{\partial H}{\partial \eta_\phi} \; = \; 0,$$

and we obtain

$$\eta_\theta \; = \; \lambda_2 \sin\left(\frac{z_3}{2}\right), \;\; \eta_\phi \; = \; 4\lambda_4 \sin\left(\frac{z_3}{2}\right). \tag{20}$$

Plugging the optimal control back into the state and costate dynamics, we obtain the following set of equation when $\sin\left(\frac{\phi}{2}\right) \neq 0$:

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -z_2 z_4 \cot\left(\frac{z_3}{2}\right) \; + \; \lambda_2 \sin^2\left(\frac{z_3}{2}\right), \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= z_2^2 \sin(z_3) \; + \; 16\lambda_4 \sin^2\left(\frac{z_3}{2}\right), \\
\dot{\lambda}_1 &= 0, \\
\dot{\lambda}_2 &= -\lambda_1 \; + \; \lambda_2 z_4 \cot\left(\frac{z_3}{2}\right) \; - \; 2\lambda_4 z_2 \sin(z_3), \\
\dot{\lambda}_3 &= -\frac{\lambda_2}{2}\left[z_2 z_4 \csc^2\left(\frac{z_3}{2}\right) \; + \; \frac{1}{2}\lambda_2 \sin(z_3)\right] \\
&- \; \lambda_4 \left[z_2^2 \cos(z_3) \; + \; 4\lambda_4 \sin(z_3)\right], \\
\dot{\lambda}_4 &= \lambda_2 z_2 \cot\left(\frac{z_3}{2}\right) \; - \; \lambda_3.
\end{aligned} \tag{21}$$

Analogously we can define state vector $\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ and costate variable vector $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)$ and define a Hamiltonian function $\bar{H}(\bar{z}, \bar{\lambda})$ as follows:

$$\bar{H}(\bar{z}, \bar{\lambda}) \; = \; \bar{\lambda} \dot{\bar{z}}^T \; - \; \frac{1}{2}(\eta_\theta^2 \; + \; \eta_\phi^2).$$

We write down the Hamiltonian equation analogous to (18) and the optimal control is obtained analogously as

$$\eta_\theta \; = \; \bar{\lambda}_2 \cos\left(\frac{\bar{z}_3}{2}\right), \;\; \eta_\phi \; = \; 4\bar{\lambda}_4 \cos\left(\frac{\bar{z}_3}{2}\right). \tag{22}$$

Plugging the optimal control back into the state and costate dynamics, we obtain the following set of equation when $\cos\left(\frac{\phi}{2}\right) \neq 0$:

$$\begin{aligned}
&\dot{\bar{z}}_1 = \bar{z}_2 \tan\left(\frac{\bar{z}_3}{2}\right), \\
&\dot{\bar{z}}_2 = -\bar{z}_2 \bar{z}_4 \left[1 \; + \; \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right] \; + \; \bar{\lambda}_2 \cos^2\left(\frac{\bar{z}_3}{2}\right), \\
&\dot{\bar{z}}_3 = \bar{z}_4 \tan\left(\frac{\bar{z}_3}{2}\right), \\
&\dot{\bar{z}}_4 = \bar{z}_2^2 \sin(\bar{z}_3) \tan\left(\frac{\bar{z}_3}{2}\right) \; - \; \frac{\bar{z}_4^2}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right) \\
&\quad + 16\bar{\lambda}_4 \cos^2\left(\frac{\bar{z}_3}{2}\right), \\
&\dot{\bar{\lambda}}_1 = 0, \\
&\dot{\bar{\lambda}}_2 = -\bar{\lambda}_1 \tan\left(\frac{\bar{z}_3}{2}\right) \; + \; \bar{\lambda}_2 \bar{z}_4 \left[1 \; + \; \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right] \\
&\quad - 2\bar{\lambda}_4 \bar{z}_2 \sin(\bar{z}_3) \tan\left(\frac{\bar{z}_3}{2}\right), \\
&\dot{\bar{\lambda}}_3 = -\frac{1}{2}\bar{\lambda}_1 \bar{z}_2 \sec^2\left(\frac{\bar{z}_3}{2}\right) \; - \; \frac{1}{2}\bar{\lambda}_3 \bar{z}_4 \sec^2\left(\frac{\bar{z}_3}{2}\right) \\
&\quad + \frac{\bar{\lambda}_2}{2}\left[\bar{z}_2 \bar{z}_4 \sec^2\left(\frac{\bar{z}_3}{2}\right) \tan\left(\frac{\bar{z}_3}{2}\right) \; + \; \left(\frac{\bar{\lambda}_2}{2}\right) \sin(\bar{z}_3)\right] \\
&\quad - \bar{\lambda}_4 \left[\bar{z}_2^2 \sin(\bar{z}_3) \; - \; \frac{1}{2}\bar{z}_4^2 \sin\left(\frac{\bar{z}_3}{2}\right) \sec^3\left(\frac{\bar{z}_3}{2}\right)\right] \\
&\quad - \; 4\bar{\lambda}_4 \sin(\bar{z}_3), \\
&\dot{\bar{\lambda}}_4 = \bar{\lambda}_2 \bar{z}_2 \left[1 \; + \; \frac{1}{2}\sec^2\left(\frac{\bar{z}_3}{2}\right)\right] \; - \; \bar{\lambda}_3 \tan\left(\frac{\bar{z}_3}{2}\right) \\
&\quad + \bar{\lambda}_4 \bar{z}_4 \sec^2\left(\frac{\bar{z}_3}{2}\right).
\end{aligned} \tag{23}$$

Note that we only have the initial and final values of the state vector $z$ or $\bar{z}$. Boundary values of $\lambda$ or $\bar{\lambda}$ are unknown. One would therefore require to solve the dynamics (21) or (23) as a Two Point Boundary Value Problem, on the chart $\sin\left(\frac{\phi}{2}\right) \neq 0$ or $\cos\left(\frac{\phi}{2}\right) \neq 0$ respectively. It is worthwhile to remark that in the two charts, the state and the costate variables $(z, \lambda)$ and $(\bar{z}, \bar{\lambda})$ are different. It is a tedious calculation to show that they are related as follows:

$$\begin{aligned}
\bar{z}_1 &= z_1, \;\; \bar{z}_3 \; = \; z_3, \;\; \bar{\lambda}_1 \; = \; \lambda_1, \\
\bar{z}_2 &= \frac{z_2}{\tan\frac{z_3}{2}}, \;\; \bar{z}_4 \; = \; \frac{z_4}{\tan\frac{z_3}{2}}, \\
\bar{\lambda}_2 &= \lambda_2 \tan\frac{z_3}{2}, \;\; \bar{\lambda}_4 \; = \; \lambda_4 \tan\frac{z_3}{2},
\end{aligned} \tag{24}$$

$$\bar{\lambda}_3 \; = \; \lambda_3 \; + \; \frac{\lambda_2 z_2 \; + \; \lambda_4 z_4}{\sin(z_3)}. \tag{25}$$

In order to derive (24), we equate (20) and (22). The rest of the coordinate transformations are obtained by taking the time derivative of (24) and simplifying the expressions. The details are omitted.

## 5 Equation of motion on SO(3)

Using the Riemannian metric (4) on SO(3), one obtains the following expression for the Lagrangian

$$\begin{aligned}
&L\left(\theta, \phi, \alpha, \dot{\theta}, \dot{\phi}, \dot{\alpha}\right) \\
&= \frac{1}{2}\left[\sin^2\left(\frac{\phi}{2}\right)\cos^2(\alpha)\dot{\theta}^2 + \sin^2\left(\frac{\phi}{2}\right)\dot{\alpha}^2 + \frac{1}{4}\dot{\phi}^2\right],
\end{aligned} \tag{26}$$

where we assume, as before, that the potential function term is absent. The Euler Lagrange's Equation (12) gives rise to the following

$$\begin{aligned}
\ddot{\theta} &= -\cot\left(\frac{\phi}{2}\right)\dot{\theta}\dot{\phi} \;+\; 2\tan(\alpha)\dot{\theta}\dot{\alpha} \;+\; \csc^2\left(\frac{\phi}{2}\right)\sec^2(\alpha)\tau_\theta, \\
\ddot{\phi} &= \sin(\phi)\left[(\cos^2(\alpha)\dot{\theta}^2 \;+\; \dot{\alpha}^2\right] \;+\; 4\tau_\phi, \\
\ddot{\alpha} &= -\frac{\sin(2\alpha)}{2}\dot{\theta}^2 \;-\; \cot\left(\frac{\phi}{2}\right)\dot{\phi}\dot{\alpha} \;+\; \tau_\alpha\csc^2\left(\frac{\phi}{2}\right).
\end{aligned} \tag{27}$$

Note that the geodesic equation (6) can be easily derived from (27) by setting the generalized control torques to be zero. Like in section 3, we define new generalized inputs $\eta_\theta$, $\eta_\phi$, and $\eta_\alpha$ as follows:

$$\begin{aligned}
\tau_\theta &= \eta_\theta \sin^3\left(\frac{\phi}{2}\right)\cos^2(\alpha), \\
\tau_\phi &= \eta_\phi \sin\left(\frac{\phi}{2}\right)\cos^2(\alpha), \\
\tau_\alpha &= \eta_\alpha \sin^2\left(\frac{\phi}{2}\right)\cos^2(\alpha).
\end{aligned}$$

The equations of motion (27) can now be written on four charts that cover SO(3), as follows:

**Chart I** ($\sin\frac{\phi}{2} \neq 0, \cos\alpha \neq 0$): State variables chosen are

$$z_1 = \theta, z_2 = \dot{\theta}, z_3 = \phi, z_4 = \dot{\phi}, z_5 = \alpha, z_6 = \dot{\alpha}.$$

The equation of motion (27) is given by

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= 2z_6z_2\tan(z_5) - z_2z_4\cot\left(\frac{z_3}{2}\right) + \eta_\theta\sin\left(\frac{z_3}{2}\right), \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= \sin(z_3)\left[z_2^2\cos^2(z_5) + z_6^2\right] + 4\eta_\phi\sin\left(\frac{z_3}{2}\right)\cos^2 z_5 \\
\dot{z}_5 &= z_6, \\
\dot{z}_6 &= -z_4z_6\cot\left(\frac{z_3}{2}\right) - \left(\frac{z_2^2}{2}\right)\sin(2z_5) + \eta_\alpha\cos^2(z_5).
\end{aligned} \tag{28}$$

**Chart II** ($\cos\frac{\phi}{2} \neq 0, \cos\alpha \neq 0$): State variables chosen are

$$z_1 = \theta, z_2 = \frac{\dot{\theta}}{\tan\frac{\phi}{2}}, z_3 = \phi, z_4 = \frac{\dot{\phi}}{\tan\frac{\phi}{2}}, z_5 = \alpha, z_6 = \dot{\alpha}.$$

The equation of motion (27) is given by

$$\begin{aligned}
\dot{z}_1 &= z_2\tan\left(\frac{z_3}{2}\right), \\
\dot{z}_2 &= 2z_2z_6\tan(z_5) - z_2z_4\left[1 + \frac{1}{2}\sec^2\left(\frac{z_3}{2}\right)\right] \;+\; \eta_\theta\cos\left(\frac{z_3}{2}\right), \\
\dot{z}_3 &= z_4\tan\left(\frac{z_3}{2}\right), \\
\dot{z}_4 &= 2z_2^2\sin^2\left(\frac{z_3}{2}\right)\cos^2(z_5) + 2z_6^2\cos^2\left(\frac{z_3}{2}\right) \\
&\quad - \left(\frac{z_4^2}{2}\right)\sec^2\left(\frac{z_3}{2}\right) + \; 4\eta_\phi\cos\left(\frac{z_3}{2}\right)\cos^2(z_5), \\
\dot{z}_5 &= z_6, \\
\dot{z}_6 &= -z_4z_6 - \left(\frac{z_2^2}{2}\right)\sin(2z_5)\tan^2\left(\frac{z_3}{2}\right) + \eta_\alpha\cos^2(z_5).
\end{aligned} \tag{29}$$

**Chart III** ($\sin\frac{\phi}{2} \neq 0, \sin\alpha \neq 0$): State variables chosen are

$$z_1 = \theta, z_2 = \dot{\theta}, z_3 = \phi, z_4 = \dot{\phi}, z_5 = \alpha, z_6 = \frac{\dot{\alpha}}{\cot\alpha}.$$

The equation of motion (27) is given by

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= 2z_2z_6 \;-\; z_2z_4\cot\left(\frac{z_3}{2}\right) \;+\; \eta_\theta\sin\left(\frac{z_3}{2}\right), \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= \sin(z_3)\left[z_2^2\cos^2(z_5) \;+\; z_6^2\cot^2(z_5)\right] \\
&\quad + \; 4\eta_\phi\sin\left(\frac{z_3}{2}\right)\cos^2 z_5, \\
\dot{z}_5 &= z_6\cot(z_5), \\
\dot{z}_6 &= -z_4z_6\cot\left(\frac{z_3}{2}\right) \;-\; z_2^2\sin^2(z_5) \;+\; z_6^2\csc^2(z_5) \\
&\quad + \; \frac{1}{2}\sin(2z_5)\eta_\alpha.
\end{aligned} \tag{30}$$

**Chart IV** ($\cos\frac{\phi}{2} \neq 0, \sin\alpha \neq 0$): State variables chosen are

$$z_1 = \theta, z_2 = \frac{\dot{\theta}}{\tan\frac{\phi}{2}}, z_3 = \phi, z_4 = \frac{\dot{\phi}}{\tan\frac{\phi}{2}}, z_5 = \alpha, z_6 = \frac{\dot{\alpha}}{\cot\alpha}.$$

The equation of motion (27) is given by

$$\begin{aligned}
\dot{z}_1 &= z_2\tan\left(\frac{z_3}{2}\right), \\
\dot{z}_2 &= 2z_2z_6 \;-\; z_2z_4\left[1 + \frac{1}{2}\sec^2\left(\frac{z_3}{2}\right)\right] \;+\; \eta_\theta\cos\left(\frac{z_3}{2}\right), \\
\dot{z}_3 &= z_4\tan\left(\frac{z_3}{2}\right), \\
\dot{z}_4 &= 2z_2^2\sin^2\left(\frac{z_3}{2}\right)\cos^2(z_5) \;+\; 2z_6^2\cos^2\left(\frac{z_3}{2}\right)\cot^2(z_5) \\
&\quad - \left(\frac{z_4^2}{2}\right)\sec^2\left(\frac{z_3}{2}\right) + 4\eta_\phi\cos\left(\frac{z_3}{2}\right)\cos^2(z_5), \\
\dot{z}_5 &= z_6\cot(z_5), \\
\dot{z}_6 &= -z_4z_6 \;+\; z_2^2\tan^2\left(\frac{z_3}{2}\right)\sin^2(z_5) \;+\; z_6^2\csc^2(z_5) \\
&\quad + \; \frac{1}{2}\eta_\alpha\sin(2z_5).
\end{aligned} \tag{31}$$

## 6 Optimal control of eye movement on SO(3)

As in section 4, a standard optimal control problem on SO(3) would be to steer the state $(\theta, \dot{\theta}, \phi, \dot{\phi}, \alpha, \dot{\alpha})$ from $(\theta_0, 0, \phi_0, 0, \alpha_0, 0)$ to $(\theta_1, 0, \phi_1, 0, \alpha_1, 0)$ in $T$ units of time while minimizing the control energy

$$\frac{1}{2}\int_0^T \left(\eta_\theta^2 \;+\; \eta_\phi^2 \;+\; \eta_\alpha^2\right) dt. \tag{32}$$

One way to handle the optimal control problem is to define state vector $z = (z_1, z_2, z_3, z_4, z_5, z_6)$ and costate variable vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ and define, as in (18), a Hamiltonian function $H(z, \lambda)$ as follows:

$$H(z, \lambda) \;=\; \lambda\dot{z}^T \;-\; \frac{1}{2}(\eta_\theta^2 \;+\; \eta_\phi^2 \;+\; \eta_\alpha^2). \tag{33}$$

We can now proceed by writing down the Hamiltonian equation, as in (19), and obtain the optimal control together with the state/costate equation, one for each chart. To economize on space, we write down the costate equation for chart 2 only.

**Chart II** ($\cos\frac{\phi}{2} \neq 0,\ \cos\alpha \neq 0$): The costate equations are given by

$$\begin{aligned}
\dot\lambda_1 &= 0, \\
\dot\lambda_2 &= -\lambda_1 \tan\left(\frac{z_3}{2}\right) - 2\lambda_2 z_6 \tan z_5 + \lambda_2 z_4 \left[1 + \frac{1}{2}\sec^2\left(\frac{z_3}{2}\right)\right] \\
&\quad - 4\lambda_4 z_2 \sin^2\left(\frac{z_3}{2}\right)\cos^2(z_5) + \lambda_6 z_2 \sin(2z_5)\tan^2\left(\frac{z_3}{2}\right), \\
\dot\lambda_3 &= -\frac{1}{2}\lambda_1 z_2 \sec^2\left(\frac{z_3}{2}\right) + \frac{1}{2} z_2 z_4 \lambda_2 \sec^2\left(\frac{z_3}{2}\right)\tan\left(\frac{z_3}{2}\right) \\
&\quad + \frac{1}{2}\lambda_2 \sin\left(\frac{z_3}{2}\right)\eta_\theta - \frac{1}{2}\lambda_3 z_4 \sec^2\frac{z_3}{2} \\
&\quad - \lambda_4\left[z_2^2 \sin(z_3)\cos^2(z_5) - z_6^2\sin(z_3) - \frac{1}{2} z_4^2 \sec^2\left(\frac{z_3}{2}\right)\tan\left(\frac{z_3}{2}\right)\right] \\
&\quad + \eta_\phi\left[2\lambda_4\cos^2(z_5)\sin\left(\frac{z_3}{2}\right)\right] + \frac{1}{2}\lambda_6 z_2^2 \sin(2z_5)\tan\left(\frac{z_3}{2}\right)\sec^2\left(\frac{z_3}{2}\right), \\
\dot\lambda_4 &= \lambda_2 z_2\left(1 + \frac{1}{2}\sec^2\left(\frac{z_3}{2}\right)\right) - \lambda_3 \tan\left(\frac{z_3}{2}\right) + z_4\lambda_4\sec^2\left(\frac{z_3}{2}\right) + \lambda_6 z_6, \\
\dot\lambda_5 &= -2\lambda_2 z_2 z_6 \sec^2(z_5) + \lambda_6\cos(2z_5)\left[z_2^2\tan^2\frac{z_3}{2}\right] \\
&\quad + \sin(2z_5)\left[2\lambda_4\left(z_2^2\sin^2\left(\frac{z_3}{2}\right) + 2\eta_\phi\cos\left(\frac{z_3}{2}\right)\right) + \eta_\alpha\lambda_6\right], \\
\dot\lambda_6 &= -2z_2\lambda_2\tan(z_5) - 4\lambda_4 z_6\cos^2\left(\frac{z_3}{2}\right) - \lambda_5 + z_4\lambda_6.
\end{aligned} \tag{34}$$

The optimal controls are obtained as follows:

$$\eta_\theta = \lambda_2\cos\frac{z_3}{2},\ \eta_\phi = 4\lambda_4\cos\frac{z_3}{2}\cos^2(z_5),\ \eta_\alpha = \lambda_6\cos^2(z_5). \tag{35}$$

The set of state/costate equations (29), (34) have to be solved as a two point boundary value problem, since only the boundary values of the state are known. The costate equations can be analogously written for charts 1, 3 and 4. Among the four charts that cover SO(3), chart 2 is most desirable, since on this chart the dynamical equations (29), (34) are not defined when the eye is pointing backwards ($\phi = \pi$), or when the axis of rotation is perpendicular to the Listing's plane ($\alpha = \frac{\pi}{2}$). At all other points, the state and costate dynamics are well defined.

In Fig. 1, we have displayed transformation of state variables between charts together with the description of the optimal control. For charts I, III and IV, description of the state and costate dynamics have been omitted.

## 7 Conclusion

In this paper, we have illustrated how a rotational movement of the eye, can be described on SO(3) using a parametrization that uses angle variables $\alpha$, $\theta$ and $\phi$, as shown in (2). Typically, when the eye movements are not restricted by the Listing's constraint, the equation of motion is described by (27). Unfortunately, these equations have singularities, one of which is in the frontal gaze direction of the eye. We show that the entire SO(3) can be covered by $4$ charts and over each chart, the equation (27) can be described by state equations (28), (29), (30) and (31) with the property that over respective charts, the equations do not have a singularity. A typical strategy during computation would be to switch between charts, when we are close to a singularity. A new scaled form of the control signals, given by $\eta_\theta$, $\eta_\phi$ and $\eta_\alpha$ is defined, that does not have to be switched when a chart is changed. An advantage of such a choice is that the form of the cost function (32) remains the same in every chart. On each of the four charts, costate equations can be written. The state/costate equations, together define solution to the optimal control problem. These equations typically give rise to a two point boundary value problem that needs to be solved. In this paper, the costate equations have been written only for chart II. We argue that chart II is most desirable since the singularities are away from the operational region of the eye orientation and cannot be attained because of physiological limitations. Similar calculations have also been shown for a submanifold LIST of SO(3), that correspond to eye movements that satisfy the Listing's constraint.

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I

$\sin\frac{\phi}{2} \neq 0, \;\; \cos\alpha \neq 0$

$z_1, z_2, z_3, z_4, z_5, z_6,$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6.$

$\eta_\theta = \lambda_2 \sin\frac{z_3}{2}$

$\eta_\phi = 4\lambda_4 \sin\frac{z_3}{2}\cos^2 z_5$

$\eta_\alpha = \lambda_6 \cos^2 z_5$

II

$\cos\frac{\phi}{2} \neq 0, \;\; \cos\alpha \neq 0$

$z_1, \frac{z_2}{\tan\frac{z_3}{2}}, z_3, \frac{z_4}{\tan\frac{z_3}{2}}, z_5, z_6,$

$\lambda_1, \lambda_2 \tan\frac{z_3}{2}, \lambda_3 + \frac{\lambda_2 z_3 + \lambda_4 z_4}{\sin z_3},$

$\lambda_4 \tan\frac{z_3}{2}, \lambda_5, \lambda_6.$

$\eta_\theta = \bar{\lambda}_2 \cos\frac{\bar{z}_3}{2}$

$\eta_\phi = 4\bar{\lambda}_4 \cos\frac{\bar{z}_3}{2}\cos^2 \bar{z}_5$

$\eta_\alpha = \bar{\lambda}_6 \cos^2 \bar{z}_5$

III

$\sin\frac{\phi}{2} \neq 0, \;\; \sin\alpha \neq 0$

$z_1, z_2, z_3, z_4, z_5, \frac{z_6}{\cot z_5},$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4,$

$\lambda_5 - \frac{2\lambda_6}{\sin(2z_5)}, \lambda_6 \cot z_5.$

$\eta_\theta = \bar{\lambda}_2 \sin\frac{\bar{z}_3}{2}$

$\eta_\phi = 4\bar{\lambda}_4 \sin\frac{\bar{z}_3}{2}\cos^2 \bar{z}_5$

$\eta_\alpha = \frac{1}{2}\bar{\lambda}_6 \sin(2\bar{z}_5)$

IV

$\cos\frac{\phi}{2} \neq 0, \;\; \sin\alpha \neq 0$

$z_1, \frac{z_2}{\tan\frac{z_3}{2}}, z_3, \frac{z_4}{\tan\frac{z_3}{2}}, z_5, \frac{z_6}{\cot z_5},$

$\lambda_1, \lambda_2 \tan\frac{z_3}{2}, \lambda_3 + \frac{\lambda_2 z_3 + \lambda_4 z_4}{\sin z_3},$

$\lambda_4 \tan\frac{z_3}{2}, \lambda_5 - \frac{2\lambda_6}{\sin(2z_5)}, \lambda_6 \cot z_5.$

$\eta_\theta = \bar{\lambda}_2 \cos\frac{\bar{z}_3}{2}$

$\eta_\phi = 4\bar{\lambda}_4 \cos\frac{\bar{z}_3}{2}\cos^2 \bar{z}_5$

$\eta_\alpha = \frac{1}{2}\bar{\lambda}_6 \sin(2\bar{z}_5)$

Fig. 1: $\eta_\theta, \eta_\phi, \eta_\alpha$ are the optimal controls that do not change between charts. We describe the optimal controls using the same bar notation in charts II, III and IV for notational simplicity. The state and costate vectors in different charts are not the same.