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ASYMPTOTICALLY STABILIZING POTENTIAL CONTROL FOR THE EYE MOVEMENT DYNAMICS

Bijoy K. Ghosh*
Takafumi Oki, Sanath D. Kahagalage
Center for BioCybernetics and Intelligent Systems
Department of Mathematics and Statistics
Texas Tech University
Lubbock, Texas 79409-1042
Email: bijoy.ghosh@ttu.edu

Indika Wijayasinghe
Department of Mathematics and Statistics
Sam Houston State University
Huntsville, Texas, 77340
Email: ixw001@shsu.edu

ABSTRACT

In this paper, we analyze the problem of stabilizing a rotating eye movement control system satisfying the Listing's constraint. The control system is described using a suitably defined Lagrangian and written in the corresponding Hamiltonian form. We introduce a damping control and show that this choice of control asymptotically stabilizes the equilibrium point of the dynamics, while driving the state to a point of minimum total energy. The equilibrium point can be placed by appropriately locating the minimum of a potential function. The damping controller has been shown to be optimal with respect to a suitable cost function. We choose alternate forms of this cost function, by adding a term proportional to the potential energy, and synthesize stabilizing control, using numerical solution to the the well known Hamilton Jacobi Bellman equation. Using Chebyshev collocation method, the newly synthesized controller is compared with the damping control.

NOMENCLATURE

g Variable denoting the Riemannian metric.
p Costate variable given by $G(\xi)\dot{\xi}$.
q_i Coordinates of the unit quaternion vector.
G Riemannian kinematic matrix.
H Hamiltonian function.
HJB Hamilton Jacobi Bellman.
J Cost index function for the optimal control problem.
KE Kinetic energy.
L Lagrangian function.
LIST A submanifold of $SO(3)$ satisfying Listing's constraint.
M Matrix transforming the $\dot{\xi}$ to the angular velocity.
Q Orthogonal matrix, an element of $SO(3)$.
T External torque vector actuating eye rotation.
U Potential energy.
V Positive definite solution of the HJB equation.
θ, ϕ, α Three angles parameterizing $SO(3)$.
ξ State vector $(\theta, \phi)^T$ of angle variables.
τ_θ, τ_ϕ Two generalized torques in *LIST*.
ω Angular velocity vector.

INTRODUCTION

The class of problem we consider in this paper has to do with actuating the gaze directions of human eye by providing a suitable torque generated by muscles that are responsible for the rotating motion. Eye rotations, steered by neural commands to the muscles, satisfy a physiological constraint introduced by Listing [1]. It is unclear, however, what functional role does the Listing's law play when the eye is steered between two gaze directions. Likewise, it is unclear if the actuating control minimizes a suitable cost function and how the final gaze, once acquired, remains stable in spite of noise inputs to the eye control system [2]. Knowledge of the asymptotically stabilizing control

*Address all correspondence to this author.

strategies implemented by the human eye movement controller, is important in the design of robotic eyes [3] that are designed to be put on humanoid robots mimicking human eye movement.

We analyze the problem of stabilizing the pointing direction of a rigid body, such as the human eye, actuated by controllers that rotate the body. Such problems have been considered recently from a geometric point of view (see [4]). Typically the problem, considered, is to rotate the gaze direction of the human eye with control strategies derived from optimal and potential control theory. The eye rotation dynamics is obtained from a suitably defined Lagrangian (see [5]), where the dynamics is to be described using Euler Lagrange's equation. A Lagrangian typically has two components, the kinetic energy and the potential energy. The point where the potential energy attains a minimum is an equilibrium point of the dynamical system. Unless the system is suitably damped, the response around the equilibrium is oscillatory. To remove the oscillatory response, one chooses a damping controller, the effect of which is to steer the system to a point of minimum total energy. In this paper, we introduce one choice of a damping controller and compare such controllers with the one obtained from optimal control based, design methods that achieve the same goal.

In order to compute the optimal control based controller, in the interval $[0,\infty]$, one needs to solve for a suitable *value function* by solving a partial differential equation that goes by the name **Hamilton Jacobi Bellman equation**. The advantage of computing the value function is that the computed controller is stabilizing, and in comparison to the damping controller, has the state rapidly approaching the equilibrium.

THE EYE MOVEMENT PROBLEM AND LISTING'S LAW

In the mid nineteenth century, studies conducted by Listing [1] and Helmholtz [6] have claimed that the orientations of the eye are completely determined as a function of the eye's gaze direction. With the exception of occasional deviation, when the head is fixed, eye follows what is known as the **Listing's Law**. Listing (see [7]) had observed that starting from a frontal gaze, any other gaze direction the eye points to, is obtained by a rotation matrix whose axis of rotation is constrained to lie on a plane, called the Listing's plane. Consequently, the set of all orientations the eye can assume is a submanifold of **SO(3)** called **LIST**. In order to describe the submanifold **LIST**, we parameterize **SO(3)** by first considering a map between $\mathbf{S}^3$, the 3-sphere in $\mathbb{R}^4$, and **SO(3)** given by

$$rot : S^3 \to SO(3), \tag{1}$$

described as

$$\left[q_0 \; q_1 \; q_2 \; q_3 \right]^T \longmapsto Q, \tag{2}$$

where

$$Q = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2+q_2^2-q_1^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2+q_3^2-q_1^2-q_2^2 \end{bmatrix}. \tag{3}$$

A point in $\mathbf{S}^3$ can be parameterized [5], using three angle variables θ, ϕ and α as follows

$$q(\theta,\phi,\alpha) = \begin{bmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \begin{pmatrix} \cos\theta\cos\alpha \\ \sin\theta\cos\alpha \\ \sin\alpha \end{pmatrix} \end{bmatrix}, \tag{4}$$

where we assume $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$ and $\alpha \in [-\frac{\pi}{2},\frac{\pi}{2}]$. Listing's plane is defined as $\alpha = 0$. Using this parametrization, we have the following sequence of maps

$$[0,\pi]\times[0,2\pi]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \xrightarrow{q} LIST \xrightarrow{rot} SO_L(3) \xrightarrow{proj} S^2 \tag{5}$$

defined by

$$(\theta,\phi,0) \longmapsto \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\theta \\ \sin\frac{\phi}{2}\sin\theta \\ 0 \end{pmatrix} \longmapsto Q(\theta,\phi,0) \longmapsto \begin{pmatrix} \sin\theta\sin\phi \\ -\cos\theta\sin\phi \\ \cos\phi \end{pmatrix}. \tag{6}$$

In Eqn. (5), we define **LIST** to be a submanifold of $\mathbf{S}^3$ and $\mathbf{SO}_L(3)$ the corresponding submanifold of **SO(3)** (see [8] for details). For the purpose of this paper, we define the eye movement dynamics as a Lagrangian system on **LIST** in the next section.

LAGRANGIAN SYSTEM

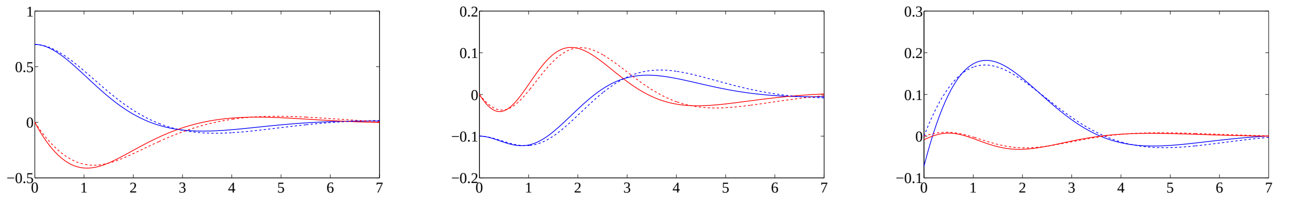
We begin by defining the state vector ξ of angle variables as $\xi = (\theta,\phi)^T$. On **LIST**, a Riemannian metric [8] is computed[1] as

$$g = \sin^2\frac{\phi}{2}\,d\theta^2 + \frac{1}{4}d\phi^2. \tag{7}$$

The metric in Eqn. (7) is induced from the standard Riemannian metric on $\mathbf{S}^3$. We write down an expression for kinetic energy given by

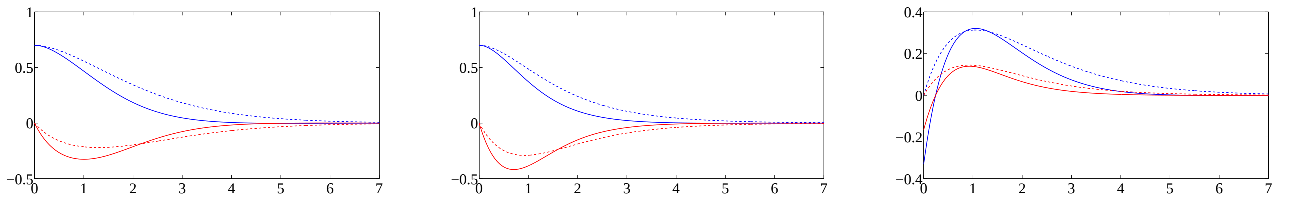
$$KE(\xi,\dot{\xi}) = \frac{1}{2}\,\dot{\xi}^T\,G(\xi)\,\dot{\xi}$$

[1] Assuming that the human eye is a perfect sphere.



(a) θ in blue, $\dot{\theta}$ in red. (b) ϕ in blue, $\dot{\phi}$ in red. (c) The optimal control $u = (u_1, u_2)^T$; u_1 in blue, u_2 in red.

FIGURE 1: Equilibrium point far from the singular point. The case $\beta_1 = \beta_2 = 1$ (solid line) is compared with the case $\beta_1 = 1$ and $\beta_2 = 0$ (dashed line). Figure shows that the effect of introducing potential function in the cost index is not significant. The initial conditions are chosen at $\theta(0) = 0.7$, $\phi(0) = 0.7$, $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) = 0$. The equilibrium values of θ and ϕ have been normalized to 0.



(a) θ in blue, $\dot{\theta}$ in red. (b) ϕ in blue, $\dot{\phi}$ in red. (c) The optimal control $u = (u_1, u_2)^T$; u_1 in blue, u_2 in red.

FIGURE 2: Equilibrium point far from the singular point. The case $\beta_1 = \beta_2 = 4$ (solid line) is compared with the case $\beta_1 = 4$ and $\beta_2 = 0$ (dashed line). The initial conditions are chosen at $\theta(0) = 0.7$, $\phi(0) = 0.7$, $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) = 0$. The equilibrium values of θ and ϕ have been normalized to 0. Figure shows that the potential function significantly hastens the time of convergence to the equilibrium.

where the matrix $G(\xi)$ is given by

$$G(\xi) = \begin{pmatrix} \sin^2 \frac{\phi}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Following [9], we write down a potential energy given by

$$U(\xi) = A(1 - q^T q^*), \tag{8}$$

where q has been described in Eqn. (6) and where q^* is a fixed point on **LIST**. The parameter A is an arbitrary constant that affects the speed of eye movement. We define a Lagrangian given by

$$L(\xi, \dot{\xi}) = KE(\xi, \dot{\xi}) - PE(\xi) = \frac{1}{2} \dot{\xi}^T G(\xi) \dot{\xi} - U(\xi),$$

with the corresponding Euler-Lagrange's equation given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} - \frac{\partial L}{\partial \xi} = u. \tag{9}$$

In Eqn. (9), vector u is the vector of generalized torques, given by $u = (\tau_\theta, \tau_\phi)^T$. We can write Eqn. (9) in a Hamiltonian form as follows

$$\dot{\xi} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial \xi} + u, \tag{10}$$

where

$$H(\xi, \dot{\xi}) = KE(\xi, \dot{\xi}) + PE(\xi) = \frac{1}{2} \dot{\xi}^T G(\xi) \dot{\xi} + U(\xi).$$

The variable p is defined as $p = G(\xi)\,\dot{\xi}$. Using variables ξ and p, we define state variable x as

$$x = \begin{pmatrix} \xi \\ p \end{pmatrix},$$

and write down the Hamilton's Eqn. (10) as

$$\dot{x} = f(x) + g(x)u. \tag{11}$$

We now consider the infinite horizon optimal control problem on the dynamical system Eqn. (11) in $\Omega \subset \mathbb{R}^n$, with equilibrium $f(0) = 0 \in \Omega$ where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $u : \mathbb{R}^n \to \mathbb{R}^m$ is the input function[2]. The optimal control problem is to find a feedback law $u(x)$ which minimizes the cost index[3]

$$J(x_0,u) = \int_0^\infty \ell(x) + u^T R u \, dt, \tag{12}$$

where $\ell : \mathbb{R}^n \to \mathbb{R}$ is positive definite, monotonically increasing and continuous and R is a symmetric, positive definite matrix. We now write the following theorem from Sepulchre, Jankovic and Kokotovic [10].

Theorem 1 (HJB): The solution to the optimal control problem of a nonlinear affine system Eqn. (11) with cost index Eqn. (12) is given by

$$u^*(x) = -\frac{1}{2} R^{-1} L_g V, \tag{13}$$

where $V(x)$ is the positive definite solution of the HJB equation

$$L_f V - \frac{1}{4} (L_g V)^T R^{-1} L_g V + \ell(x) = 0, \tag{14}$$

with boundary condition $V(0) = 0$. □

Choosing $V(x) = H(\xi, p)$, one can compute $L_f V = 0$ and $L_g V = \dot{\xi}$. The Eqn. (14) is solved as

$$\ell(x) = \frac{1}{4} \dot{\xi}^T R^{-1} \dot{\xi}.$$

When $R = c\,G^{-1}$, the cost index is given by

$$J(x_0,u) = \int_0^\infty \frac{1}{2c} KE + c\, u^T G^{-1} u \, dt, \tag{15}$$

and the optimal control is given by

$$u^* = -\frac{1}{2c} G(\xi)\,\dot{\xi}. \tag{16}$$

This was the asymptotically stabilizing, damping control considered in [9]. When $R = 4c(M^T M)^{-1}$, where M is a 3×2 matrix given by

$$M(\xi) = \begin{pmatrix} -\sin\phi\,\sin\theta & \cos\theta \\ \sin\phi\,\cos\theta & \sin\theta \\ 1-\cos\phi & 0 \end{pmatrix},$$

the cost index is given by

$$J(x_0,u) = \int_0^\infty \frac{1}{16c} \|\omega\|^2 + 4c\|T\|^2 \, dt,$$

where ω is the angular velocity vector and T is the external torque vector. The vector ω is related to the vector $\dot{\xi}$ by the relation $\omega = M(\xi)\,\dot{\xi}$. The generalized torque vector u is related to the external torque vector T as $u = M^T\,T$. The optimal control is given by

$$u^* = -\frac{1}{8c} M^T M\,\dot{\xi} = -\frac{1}{8c} M^T\,\omega, \tag{17}$$

or equivalently

$$T^* = (M^T)^{-1} u^* = -\frac{1}{8c}\,\omega. \tag{18}$$

We now claim that controllers of the form Eqns. (16), (17) are asymptotically stabilizing, with the equilibrium point given by $x = x^*$ (corresponding to $q = q^*$). The proof has been sketched in the Appendix A. Note in particular that the *angular velocity feedback* described in Eqn. (18) is independent of the choice of coordinates.

Remark 1: When the human eye is modeled as a perfect sphere, i.e when the moment of inertia matrix with respect to the body coordinate is given by $\frac{1}{4} I_{3\times3}$, one can compute that $G = \frac{1}{4} M^T M$ and the two controllers Eqns. (16), (17) are identical. For all

[2]In the problem of eye movement satisfying Listing's constraint, both n and m are 2

[3]The state dependent part of the cost function is kept sufficiently general. Subsequently, we shall choose this term to be proportional, separately, to the kinetic energy and potential energy. The controller is penalized for moving the eye very rapidly and remaining away from the final equilibrium point for too long.

other cases, the former control depends on the moment of inertia whereas the latter does not depend. □

Controllers of the form Eqns. (16), (17) are called damping controllers. They have been introduced in [9] as a class of potential controllers that drive the state from an arbitrary initial position to a desired final position, given by ξ^* the minimum point of the potential function $U(\xi)$. In this section, we have claimed that the potential controllers are optimal with respect to cost functions of the form Eqn. (12). In the next section, these cost functions are explored in details.

ASYMPTOTIC STABILIZATION VIA OPTIMAL CONTROL

In this section, we consider a cost index for optimal controller given by

$$J(x_0,u) = \int_0^\infty 2\beta_1\, KE(\xi,\dot{\xi}) + \beta_2\, U(\xi) + u^T R\, u\, dt, \quad (19)$$

where β_1 and β_2 are arbitrary constants, and the matrix R, as in Eqn. (12) is symmetric and positive definite. Note that the cost index Eqn. (15), when $R = cG^{-1}$, is a special case of Eqn. (19), wherein a term proportional to the potential energy has been explicitly added.

In order to calculate the optimal controller, u, we need to solve the HJB Eqn. (14). These equations were originally studied by Beard, Saridis and Wen [11], using a proposed method of solving Generalized Hamilton Jacobi Bellman equation (GHJB), which is a linear partial differential equation written as follows

$$\frac{\partial V}{\partial x}\,(f + gu) + \ell(x) + u^T R\, u = 0. \quad (20)$$

For a given u, the GHJB Eqn. (20) is solved for V and a new control function is constructed using Eqn. (13). This construction is continued iteratively, and Beard et al [11] have shown that the control functions converge uniformly to an optimal input where the sequence of $V(x)$-s converge uniformly to a solution of the HJB Eqn. (14). Their procedure, however, required a large number of numerical integration for computing inner products (see [12]). In this paper, we have implemented the solution of the GHJB equation using Chebyshev collocation method [13]. Details of this implementation, requiring Chebyshev interpolation, has been omitted.

In Figs. 1, 2, the minimum for the potential function is chosen as $\theta = 0$ and $\phi = \frac{\pi}{4}$. In Fig. 1, the case $\beta_1 = \beta_2 = 1$ (solid line) is compared with the case $\beta_1 = 1$ and $\beta_2 = 0$ (dashed line). This figure shows that the introduction of the potential function in the cost index Eqn. (19), does not have a significant impact. In Fig. 2, the case $\beta_1 = \beta_2 = 4$ (solid line) is compared with the case $\beta_1 = 4$ and $\beta_2 = 0$ (dashed line). This figure shows that the introduction of the potential function in the cost index does have a significant impact on the rate at which the state variables approach the equilibrium[4]. Note that the point where the dynamical system Eqn. (11) is singular is at $\phi = 0$. We would like to remark that in the simulations sketched in Figs. 1, 2, the singular point of the dynamical system is far from the chosen equilibrium point. The Chebyshev collocation method, that we have used to synthesize the optimal controller converges in the interval $[-0.7, 0.7]$ for each of the four variables $\theta, \dot{\theta}, \phi, \dot{\phi}$. Finally, note from the figures that the magnitude of the control is not significantly altered between Figs. 1, 2. Thus we make the following remark.

Remark 2: Adding a potential function to the cost index improves the performance (faster convergence to the equilibrium), without an additional cost to the control.

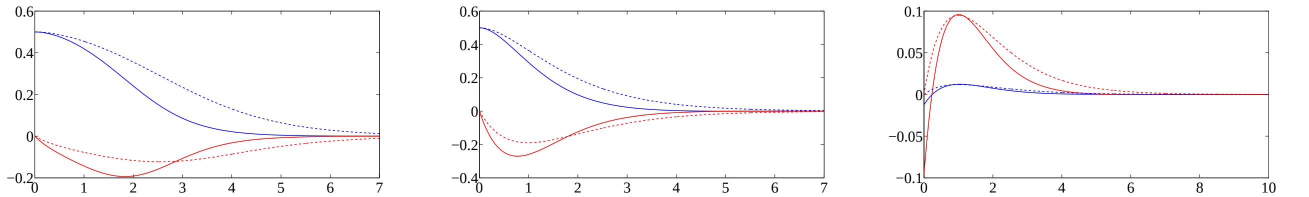
In Figs. 3, 4, we choose the point of equilibrium ($\theta = 0$ and $\phi = \frac{\pi}{16}$), close to the point of singularity. The two figures correspond to identical cost index but different initial conditions (see figure caption). The simulations indicate that the Chebyshev collocation method converges in the interval $[-0.7, 0.7]$ for each of the three variables $\theta, \dot{\theta}, \dot{\phi}$ and $[-0.1, 1.3]$ for the variable ϕ. In the neighborhood of the equilibrium point, the control u_1 is penalized higher than that of u_2 in the cost index described in Eqn. (19). As a result, the optimal u_1 control has a very small peak value in comparison to u_2 (evident from Figs. 3c, 4c). We would like to remark about one anomalous observation in Fig. 4a, where introduction of the potential function did not speed up the response.

CONCLUSION

Main idea in this paper is to study the problem of steering the human eye from an arbitrary gaze direction to a final gaze direction. The goal is to design a controller that asymptotically stabilizes the final gaze. It is widely believed that the human eye controller actively stabilizes the eye-gaze, once it acquires a fixed target in space. The precise mechanism of how asymptotic stability is achieved is unknown and optimal control strategy is introduced in this paper for the purpose.

In an earlier paper [9], we had introduced potential controllers for the purpose of steering eye-gaze direction to a point of zero potential. In any implementation of the potential control, one must add a suitable damping term. This term is chosen to be proportional to the associated velocities of the state variables, implementing damping due to friction.

[4] In the Figs. 1-4, the equilibrium point has been normalized to $x = 0$. The constant A for the potential function (8) was chosen as $A = 1$.

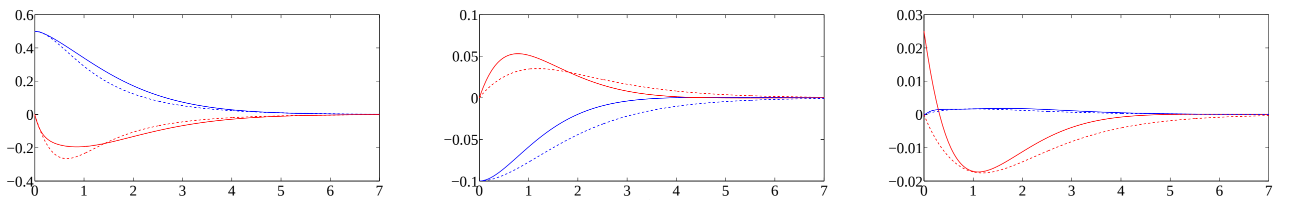


(a) θ in blue, $\dot{\theta}$ in red.

(b) ϕ in blue, $\dot{\phi}$ in red.

(c) The optimal control $u = (u_1, u_2)^T$; u_1 in blue, u_2 in red.

FIGURE 3: Equilibrium point close to the singular point. The case $\beta_1 = \beta_2 = 4$ (solid line) is compared with the case $\beta_1 = 4$ and $\beta_2 = 0$ (dashed line). The initial conditions are chosen at $\theta(0) = .5$, $\phi(0) = .5$, $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) = 0$. The equilibrium values of θ and ϕ have been normalized to 0.



(a) θ in blue, $\dot{\theta}$ in red.

(b) ϕ in blue, $\dot{\phi}$ in red.

(c) The optimal control $u = (u_1, u_2)^T$; u_1 in blue, u_2 in red.

FIGURE 4: Equilibrium point close to the singular point. The case $\beta_1 = \beta_2 = 4$ (solid line) is compared with the case $\beta_1 = 4$ and $\beta_2 = 0$ (dashed line). The initial conditions are chosen at $\theta(0) = .5$, $\phi(0) = -.1$, $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) = 0$. The equilibrium values of θ and ϕ have been normalized to 0.

In this paper we show, quite surprisingly, that the added damping can in fact be implemented as an optimal controller, where the associated cost function is described over an infinite horizon. As an example, a new damping controller is introduced, proportional to the eye angular velocity.

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Appendix A: Asymptotic Stabilization with Damping Control

We consider a Lagrange dynamical system which follows from the Euler Lagrange equation Eqn. (9), defined on an open set E. We assume that the corresponding potential function $U(\xi)$ has only one zero at $\xi = \xi^*$ and let $G(\xi)$ be the kinematic matrix. Assume that $U(\xi)$ is continuous and differentiable and $G(\xi)$ is invertible, differentiable, bounded continuous and positive definite on E. The Euler Lagrange's equation can be rewritten as

$$\ddot{\xi} = G^{-1}(\xi)\left\{-\dot{G}(\xi)\dot{\xi} - \frac{1}{2}\frac{\partial}{\partial\xi}\left(\dot{\xi}^T G(\xi)\dot{\xi}\right) - \frac{\partial}{\partial\xi}U(\xi) + u\right\}. \tag{21}$$

Without any loss of generality, let us assume that $(\xi^*, \dot{\xi}^*) = (0,0) \in E$ is an equilibrium point of Eqn. (21). We now consider the feedback control of the damping type, given by

$$u = -c\, G(\xi)\dot{\xi} \tag{22}$$

where c is a positive constant. We have the following theorem.

Theorem 2: The feedback Eqn. (22) renders the equilibrium of the system Eqn. (21) asymptotically stable.

Proof of Theorem 2: Let us define a total energy function of

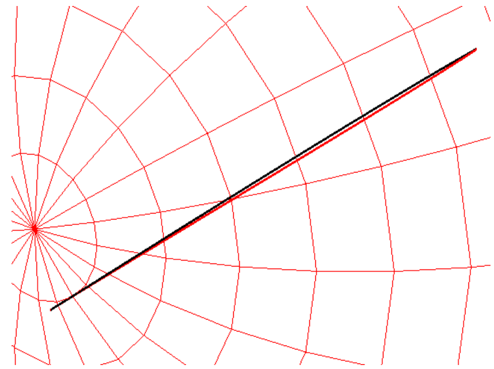

FIGURE 5: In this figure, the trajectory for the eye movement is displayed as a projection on the gaze space. The figure shows convergence to the equilibrium point $\theta = -\frac{\pi}{3}$, $\phi = -\frac{\pi}{3}$ (located at top, right) starting from the initial point $\theta = \frac{\pi}{18}$, $\phi = \frac{\pi}{18}$ (located at bottom, left). The black color is using the control in Eqn. (22). The red color is using the control in Eqn. (23).

Eqn. (21), given by

$$H(\xi,\dot{\xi}) = \frac{1}{2}\,\dot{\xi}^T G(\xi)\dot{\xi} + U(\xi).$$

Time derivative of the total energy, with input u is given by

$$\dot{H}|_{u=-cG\dot{\xi}} = -c\,\dot{\xi}^T\,G(\xi)\,\dot{\xi} \leq 0.$$

Let $\varepsilon > 0$ be a small number and let $B_\varepsilon(0)$ be an open ball around 0 of radius ε, be such that $\bar{B}_\varepsilon(0) \subset E$. Since H is continuous, we can define a number

$$\alpha = \min\left\{H(\xi,\dot{\xi}) \mid \|(\xi,\dot{\xi})\| = \varepsilon\right\}.$$

Let us pick $0 < \beta < \alpha$ and define

$$\Omega_\beta = \left\{(\xi,\dot{\xi}) \in \bar{B}_\varepsilon(0) \mid H(\xi,\dot{\xi}) \leq \beta)\right\}.$$

It follows that $\Omega_\beta \subset B_\varepsilon(0)$. Since $\dot{H} \leq 0$ on E, and Ω_β is compact, positive invariant under the dynamics Eqns. (21), (22). We now propose to use the Krasovskii-LaSalle's Theorem [14], [15].

Let S be the subset of Ω_β defined as

$$S = \left\{(\xi,\dot{\xi}) \in \Omega_\beta \mid \dot{H} = 0\right\}.$$

It would follow that S contains points in Ω_β of the form $(\xi, 0)$. We now claim that $(0,0)$ is the largest invariant set of S under the dynamical system Eqns. (21), (22). This is because, at a point $(\xi, 0)$, where $\xi \neq 0$,

$$\ddot{\xi} = -G^{-1}(\xi) \left[\frac{\partial U(\xi)}{\partial \xi}\right] \neq 0.$$

Thus the point $(\xi, 0)$ is not invariant. Using Krasovskii-LaSalle's invariance principle, $(0,0)$ is asymptotically stable under the input Eqn. (22). □

Remark 3: When the control input is of the form

$$u = -c \ M^T M(\xi)\,\dot{\xi} \tag{23}$$

as in Eqn. (17), the system Eqn. (21) is asymptotically stable as well. Proof is identical to the proof of theorem 2 in this appendix. □

Remark 4: In Fig. 5 simulated trajectories are displayed using parameters of the human eye. For this simulation, the eye is not assumed to be a perfect sphere (see [4]). The moment of inertia matrix, with respect to the body coordinates, is assumed to be

$$\mathrm{diag}(m_1 \quad m_2 \quad m_3),$$

where $m_1 = \frac{1}{5}(b^2+c^2)m$, $m_2 = \frac{1}{5}(a^2+c^2)m$, $m_3 = \frac{1}{5}(a^2+b^2)m$ and where $a = 24$ mm, $b = 25$ mm, $c = 24.5$ mm and $m = 7.5$ gm. In this case, the controllers in Eqn. (22) and Eqn. (23) are not identical[5] but both are asymptotically stabilizing the dynamics in Eqn. (21). For the control input described in (22), the coefficient A for the potential function in Eqn. (8) is chosen to be $A = 100$ and the friction coefficient c in (22) is chosen as $c = 1$. Likewise, for the control input described in (23), the coefficient A is chosen to be $A = 100$ and the friction coefficient c in (23) is chosen as $c = 650$. □

[5]Evident from the fact that the two trajectories in Fig. 5 are not identical.