Geometry and Control of Human Eye Movements

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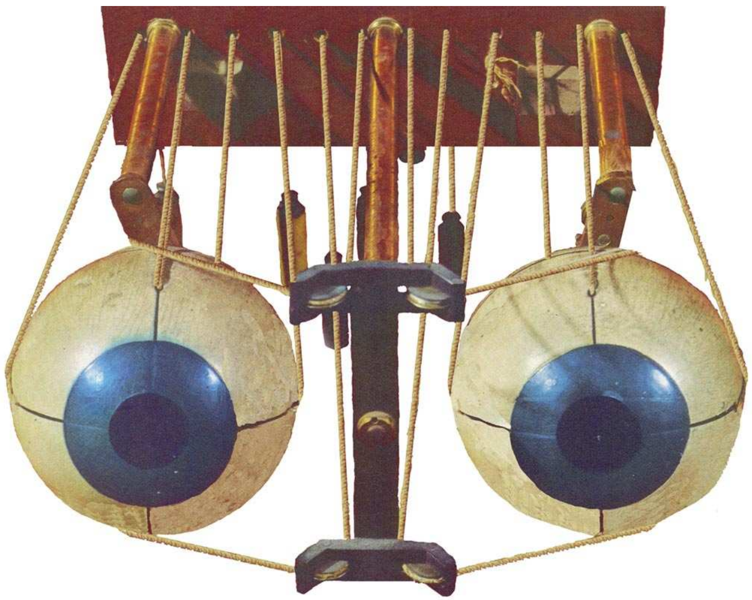

Knapp's 1861 ball and string opthalmotrop: An early mechanical model reflecting only extraocular geometry, ignoring contractile and elastic properties and coordination of inputs

Outline of the talk

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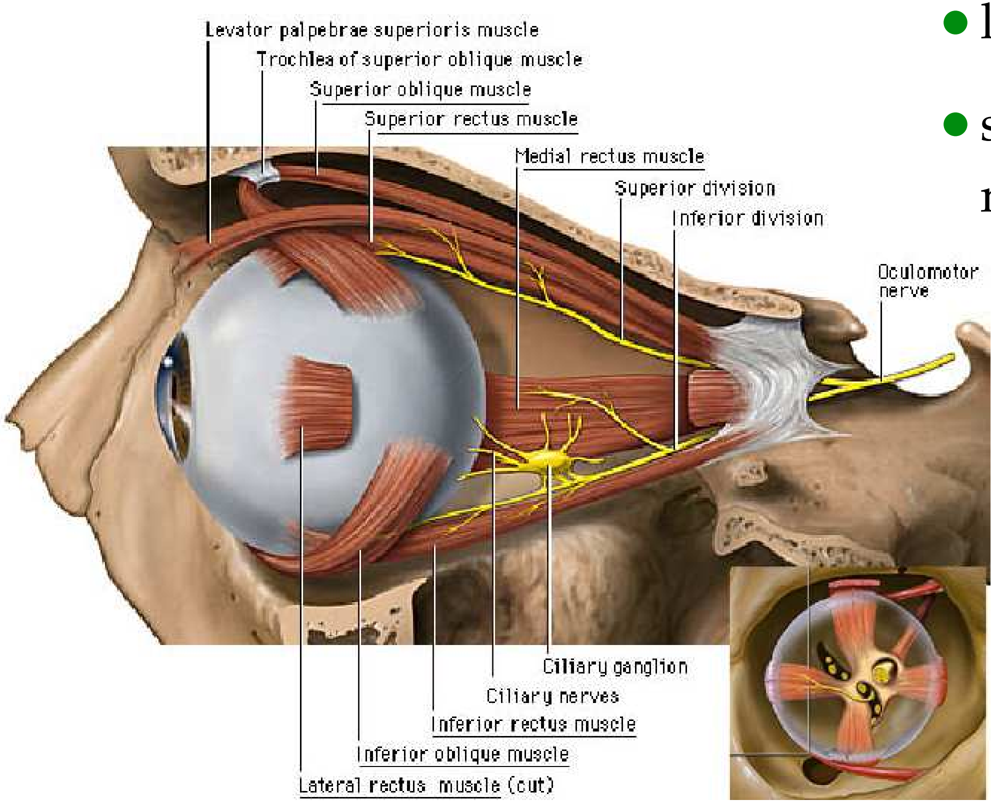
Anatomy of the Eye

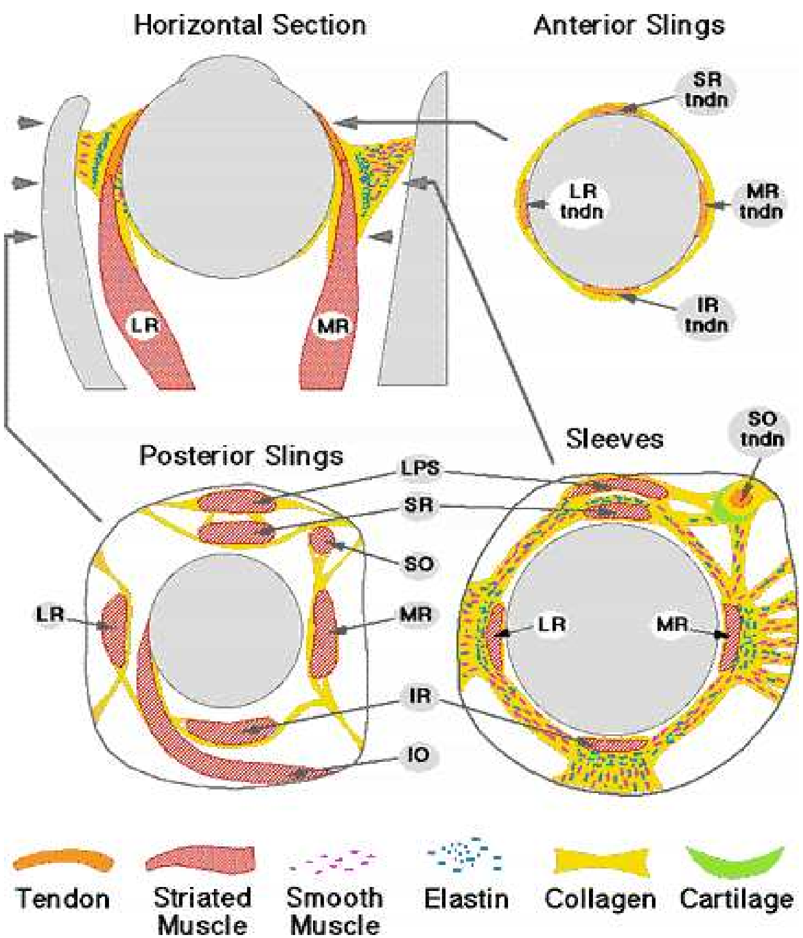


Six muscles acting as agonist/ antagonist pairs:

- superior/inferior rectus muscles
- lateral/medial rectus muscles
- superior/inferior oblique muscles

Muscle pulleys

Muscles pass through **pulleys**



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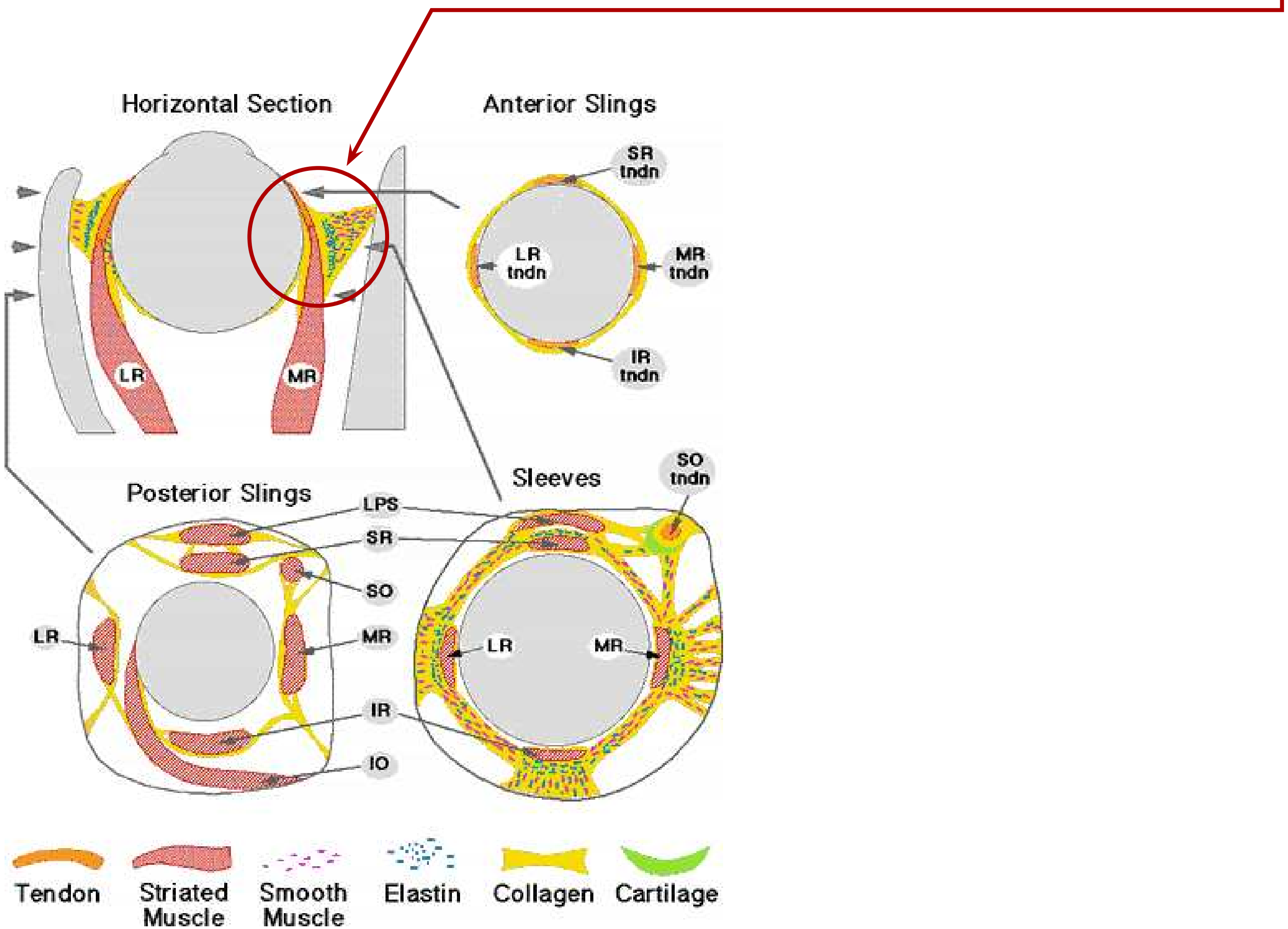
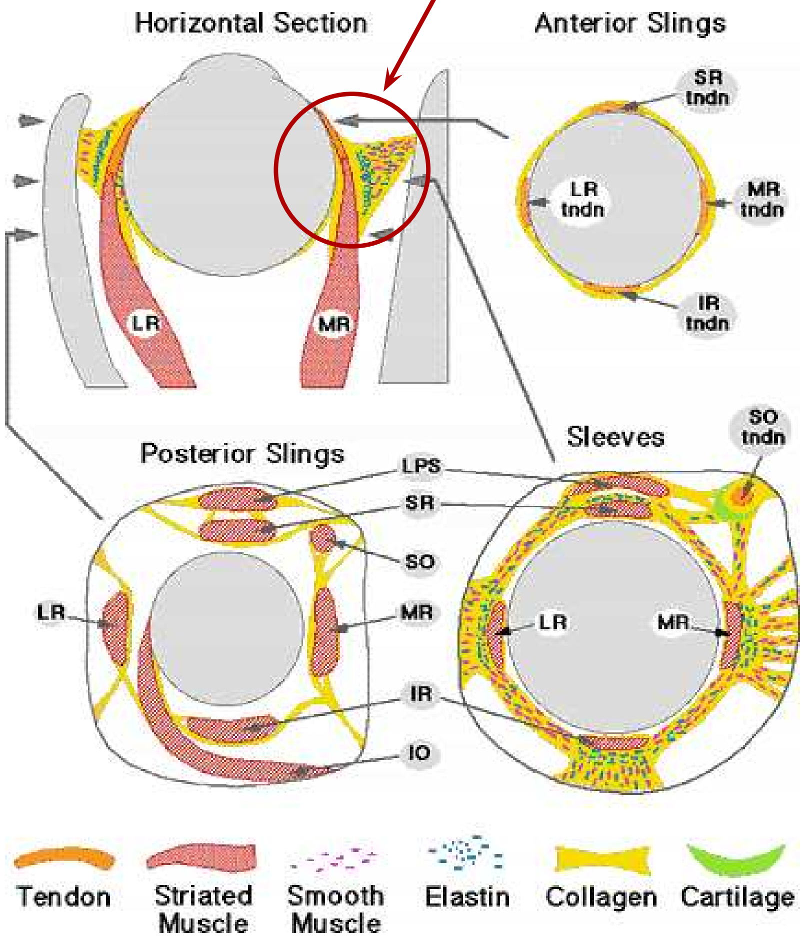


Muscle pulleys

Muscles pass through **pulleys**

- determine the pulling direction



Movements of the Eye

- **Saccades:** are the fastest eye movements (velocities: $30 \sim 700^0/s$ and lasting for about 40ms). Aim is to precisely redirect the gaze to the target to have a stabilized image on the retina (diameter of about a degree). Ex: reading, a sudden eccentric sound. Happens under open-loop control.

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- **Vestibular Ocular Reflex (VOR):** compensates for the movement of the head ensuring a clear image of the target on retina.
- **Vergence movements:** are the ones where the target moves along the gaze axis toward or away from the eye. The eye, which has the target moves along the gaze axis, remains stationary.

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Planer Eye Movements

- To simplify experiments and analysis.
- Study of planer eye movements has led to a remarkable understanding of one-dimensional movements, from the muscle mechanics to the underlying neural control system.
- A detailed biomechanical model was proposed by **Martin & Schovanec (1997)**, (**M-S model**) and studied saccadic eye movements. **Sugathadasa et al.(2000)** further investigated smooth-pursuit tracking problem.

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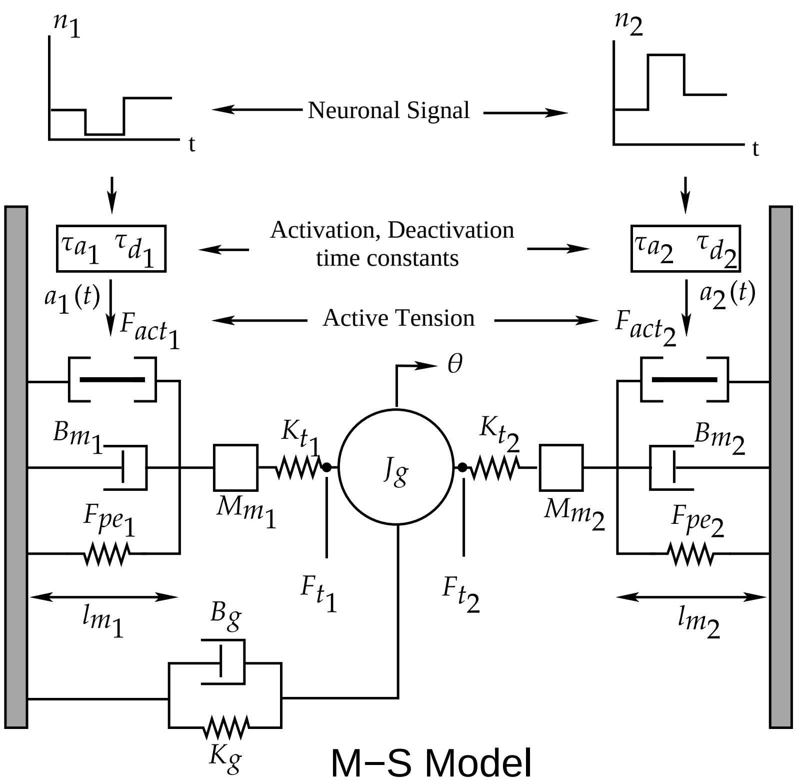
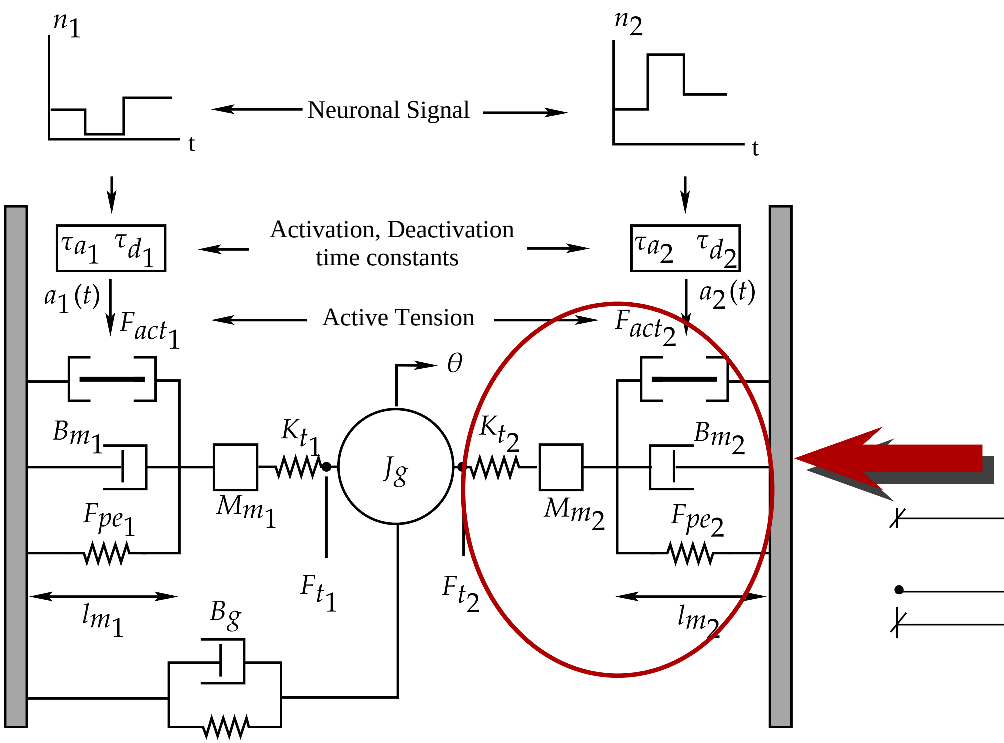
- **Polpitiya & Ghosh (2002)** proposed **"Learning Curves"** for open-loop saccadic movement control using the M-S model.

M-S Model



M−S Model

M-S Model



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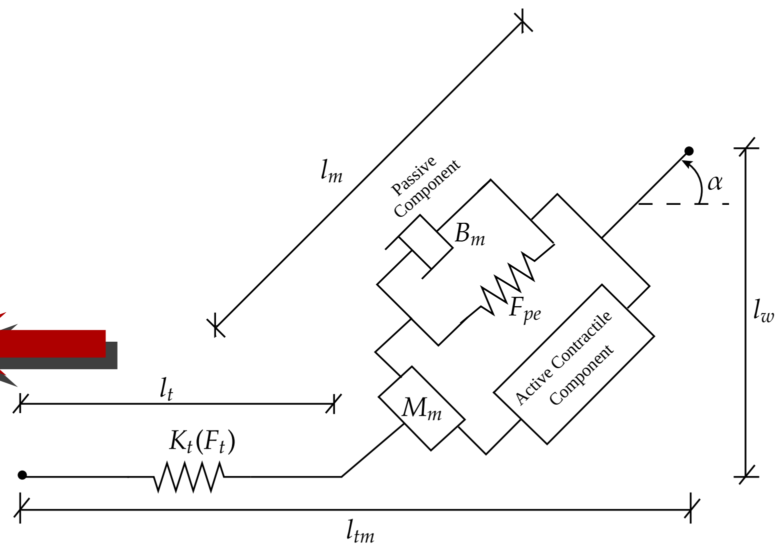
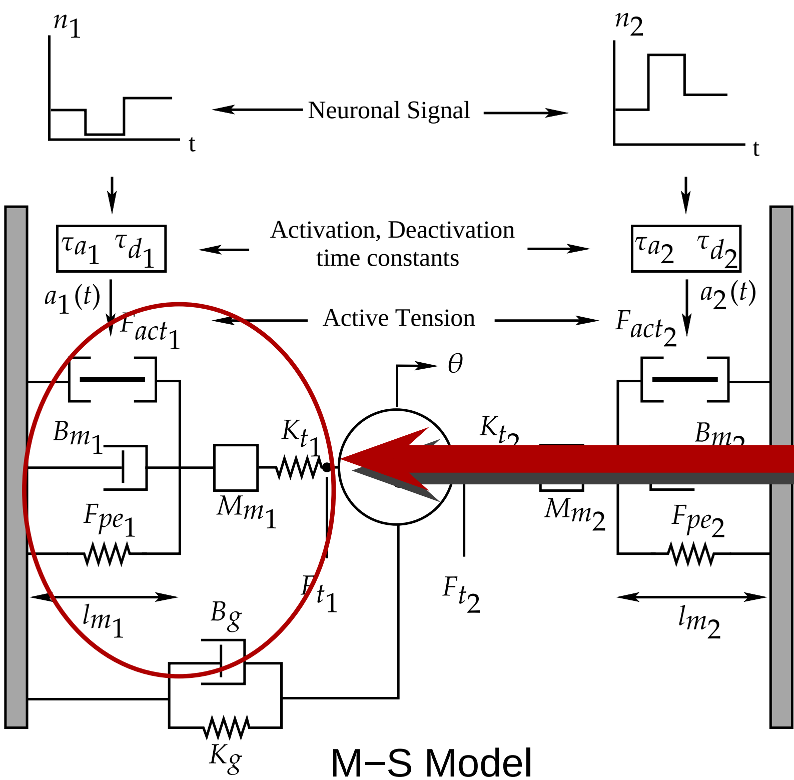


Hill-type musculotendon

M-S Model



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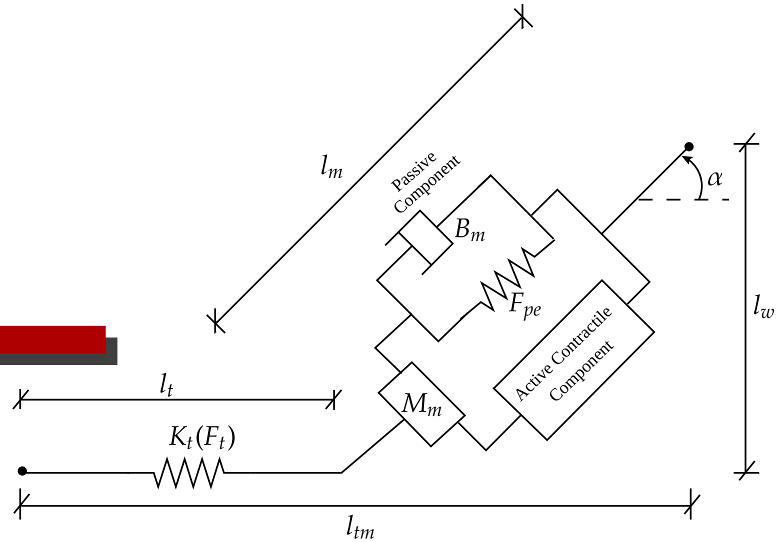

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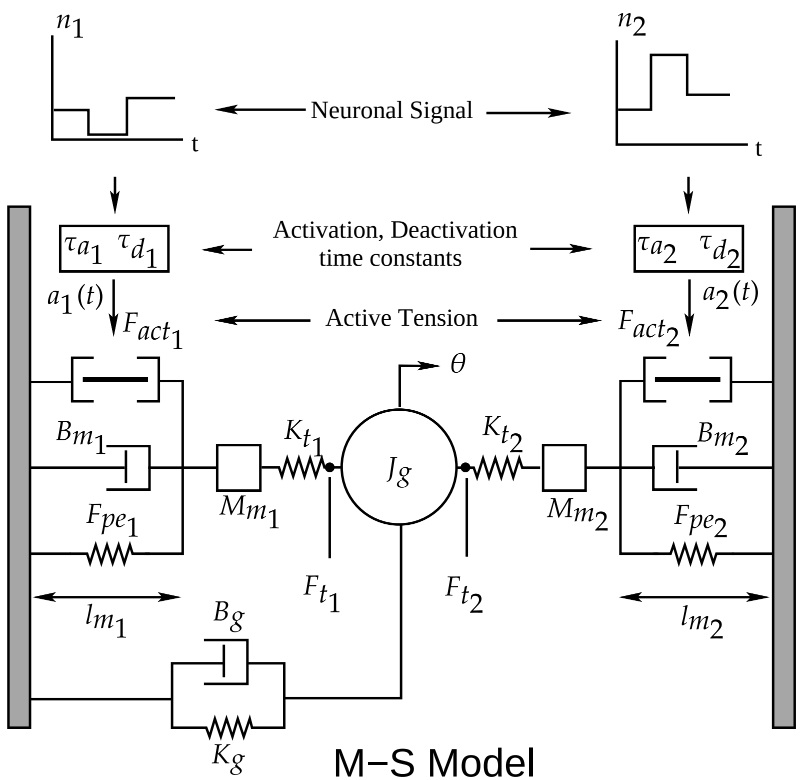

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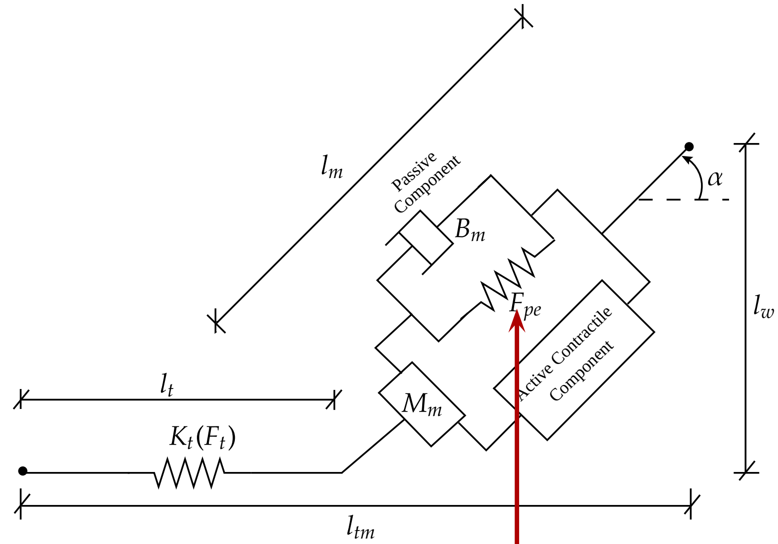
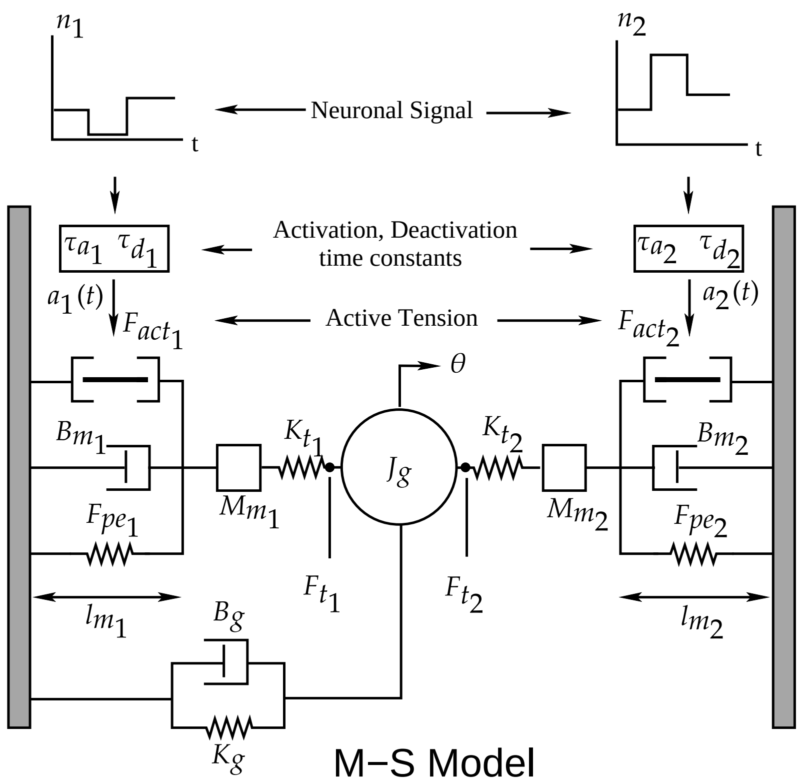


Hill-type musculotendon

$$F_{pe}(l_m) = \begin{cases} (\frac{k_{ml}}{k_{me}})[exp(k_{me}(l_m - l_{ms})) - 1] & l_{ms} \le l_m < l_{mc} \\ k_{pm}(l_m - l_{mc}) + F_{mc} & l_m > l_{mc} \\ 0 & \text{otherwise} \end{cases}$$

M-S Model



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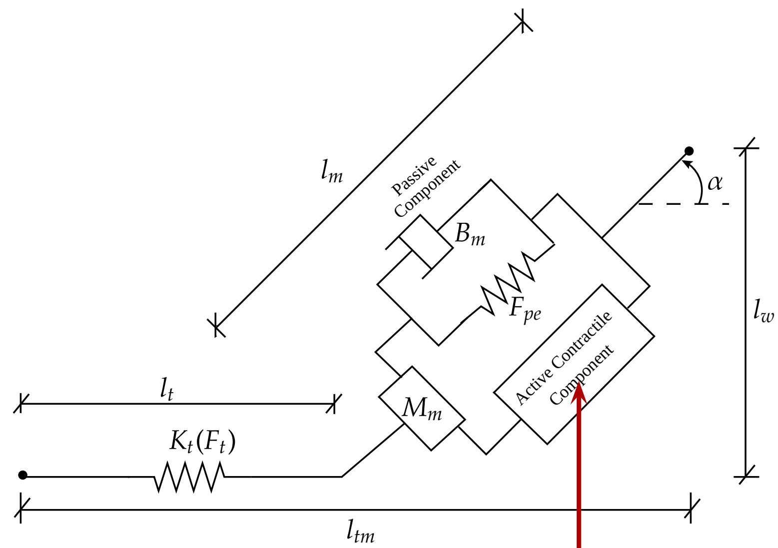
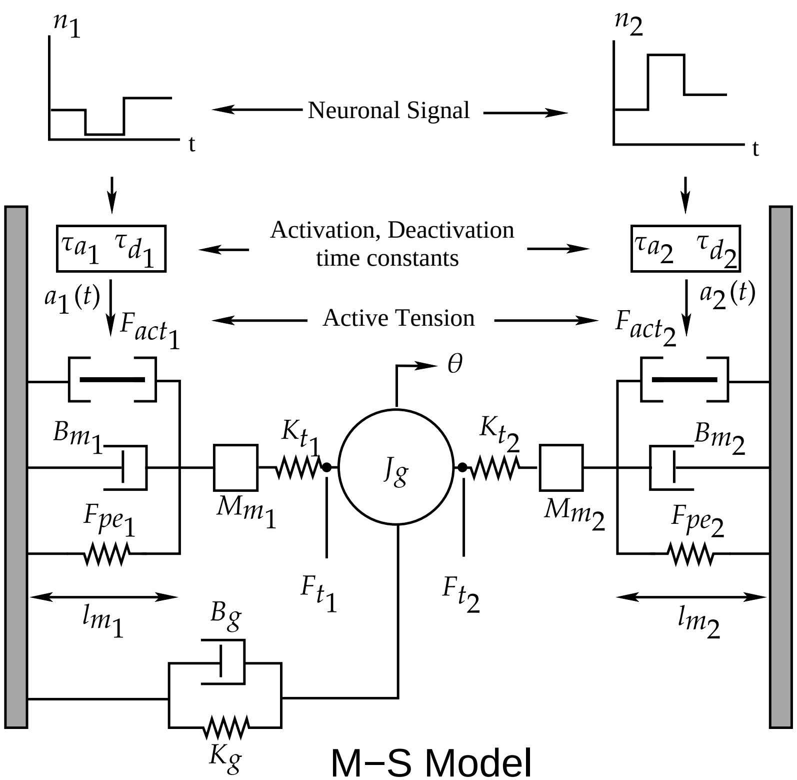


Hill-type musculotendon

$$F_{act} = F_o f_l(\tilde{l}_m) f_v(\tilde{v}_m) \times a(t)$$

where $\tilde{v}_m = \dot{\tilde{l}}_m$.

M-S Model



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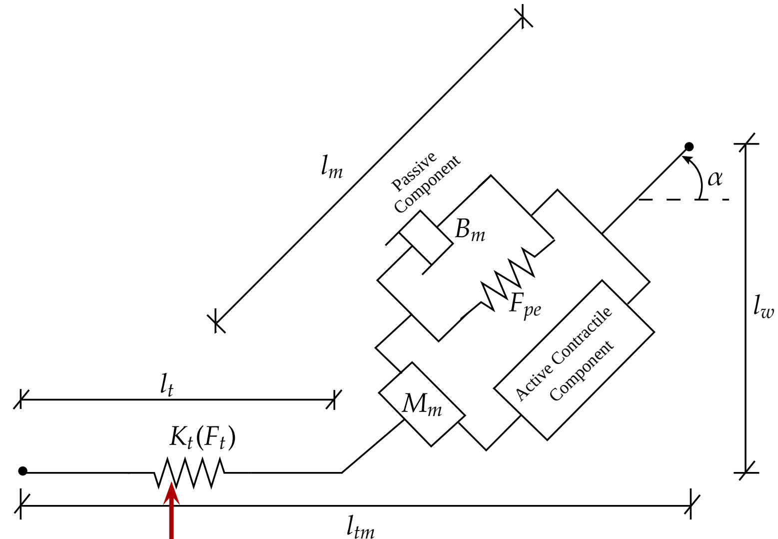

Hill-type musculotendon

$\dot{F}_t = K_t(F_t)\dot{l}_t$ where

$$K_t(F_t) = \begin{cases} k_{te}F_t + K_{tl}, & 0 \leq F_t < F_{tc} \\ k_s, & F_t \geq F_{tc} \end{cases}$$

M-S Model

The equation of motion for the eye globe can be written as:

$$J_g\ddot{\theta} + B_g\dot{\theta} + K_g\theta = F_{t_1} - F_{t_2}$$

where J_G, B_G, and K_G denote the globe inertia, globe viscosity, and globe elasticity. J_g, B_g, K_g are obtained as $J_g = \frac{J_G}{980r(180/\pi)}$ with r denoting the radius of the eye globe.

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Model can be written in the form $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$ where u_1 and u_2 are the neural inputs, let the state vector be $x^T(t) = [\theta, \dot{\theta}, l_{m1}, \dot{l}_{m1}, l_{m2}, \dot{l}_{m2}, F_{t_1}, F_{t_2}, a_1, a_2]$.

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$$f(x) = \begin{bmatrix} x_2 \\ \frac{1}{J_g}(x_7 - x_8 - B_g x_2 - K_g x_1) \\ x_4 \\ \frac{980}{M}(x_7 - F_{act}(x_3, x_4, x_9) - F_{pe}(x_3) - B_{pm}(\frac{180}{\pi r})x_4) \\ x_6 \\ \frac{980}{M}(x_8 - F_{act}(x_5, x_6, x_{10}) - F_{pe}(x_5) - B_{pm}(\frac{180}{\pi r})x_6) \\ K_t(x_7)\left[-x_2 - (\frac{180}{\pi r})x_4\right] \\ K_t(x_8)\left[x_2 - (\frac{180}{\pi r})x_6\right] \\ -\frac{x_9}{\tau_1} \\ -\frac{x_{10}}{\tau_1} \end{bmatrix}$$

$$g_1(x) = (1/\tau_1)[0,0,0,0,0,0,0,0,1,0]$$

$$g_2(x) = (1/\tau_1)[0,0,0,0,0,0,0,0,0,1].$$

M-S Model: Simulations

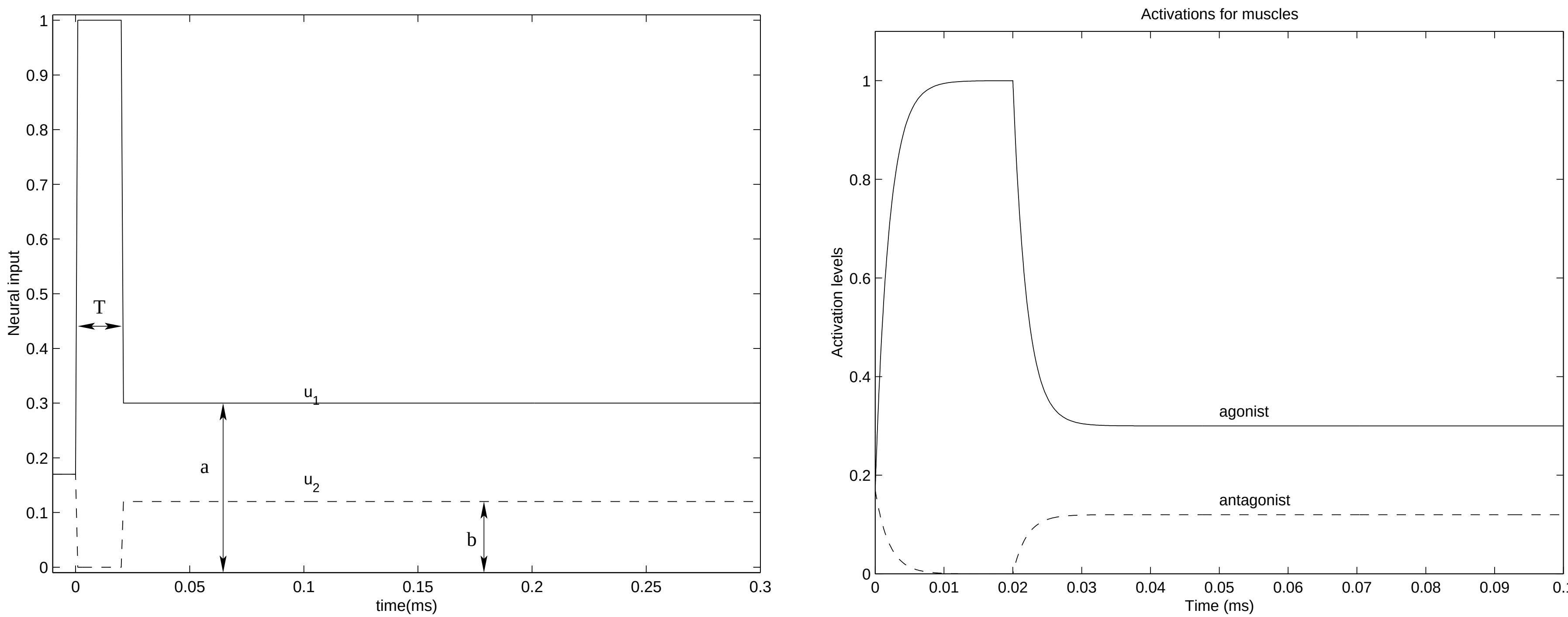


Figure 1: Neuronal inputs and the resulting activation signals to the agonist and antagonist (10^0 saccade)

M-S Model: Simulations

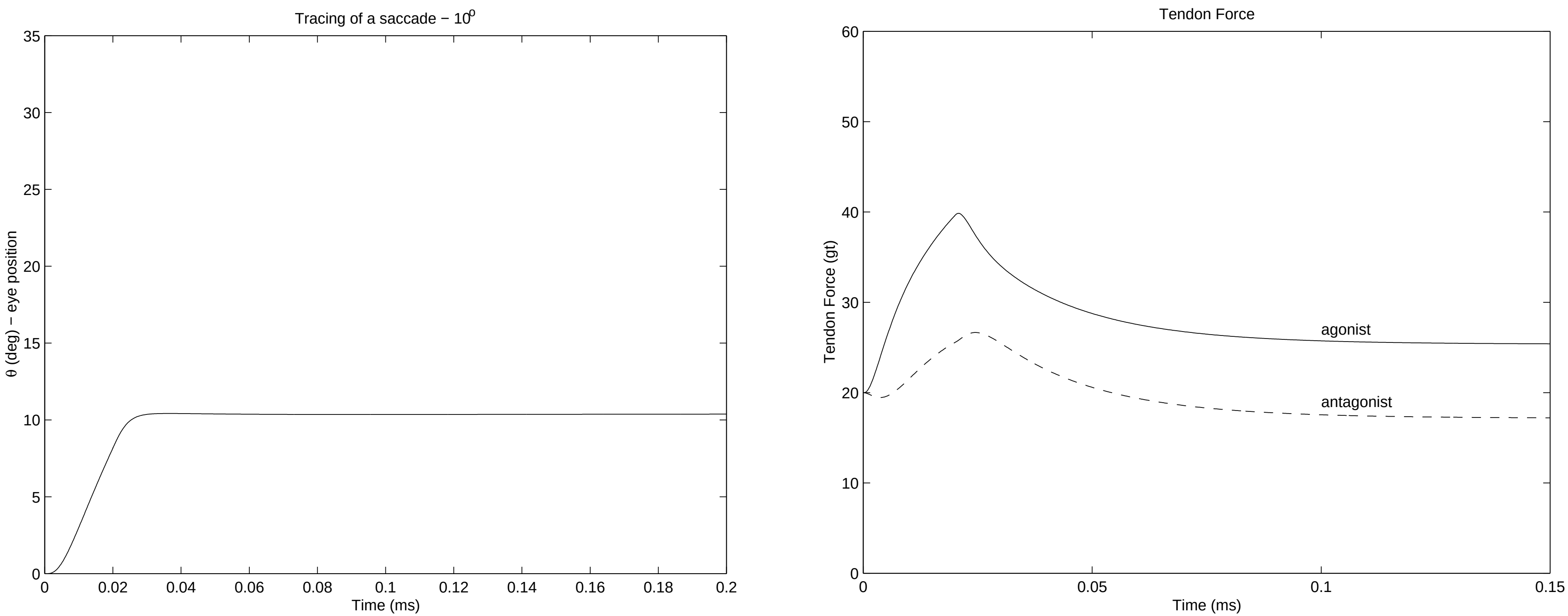


Figure 2: Simulation of 10^o Saccade and the corresponding forces in the tendons

Learning Curves

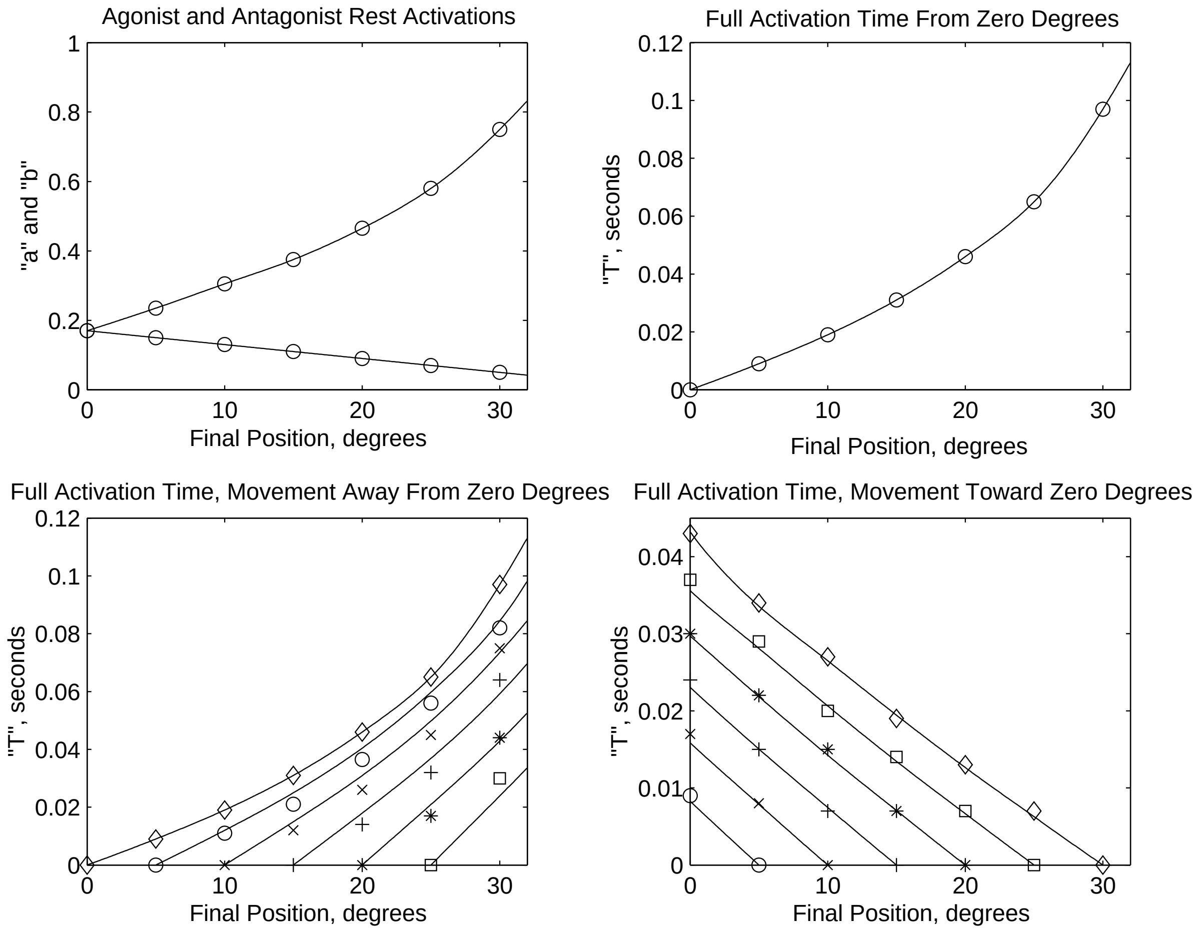
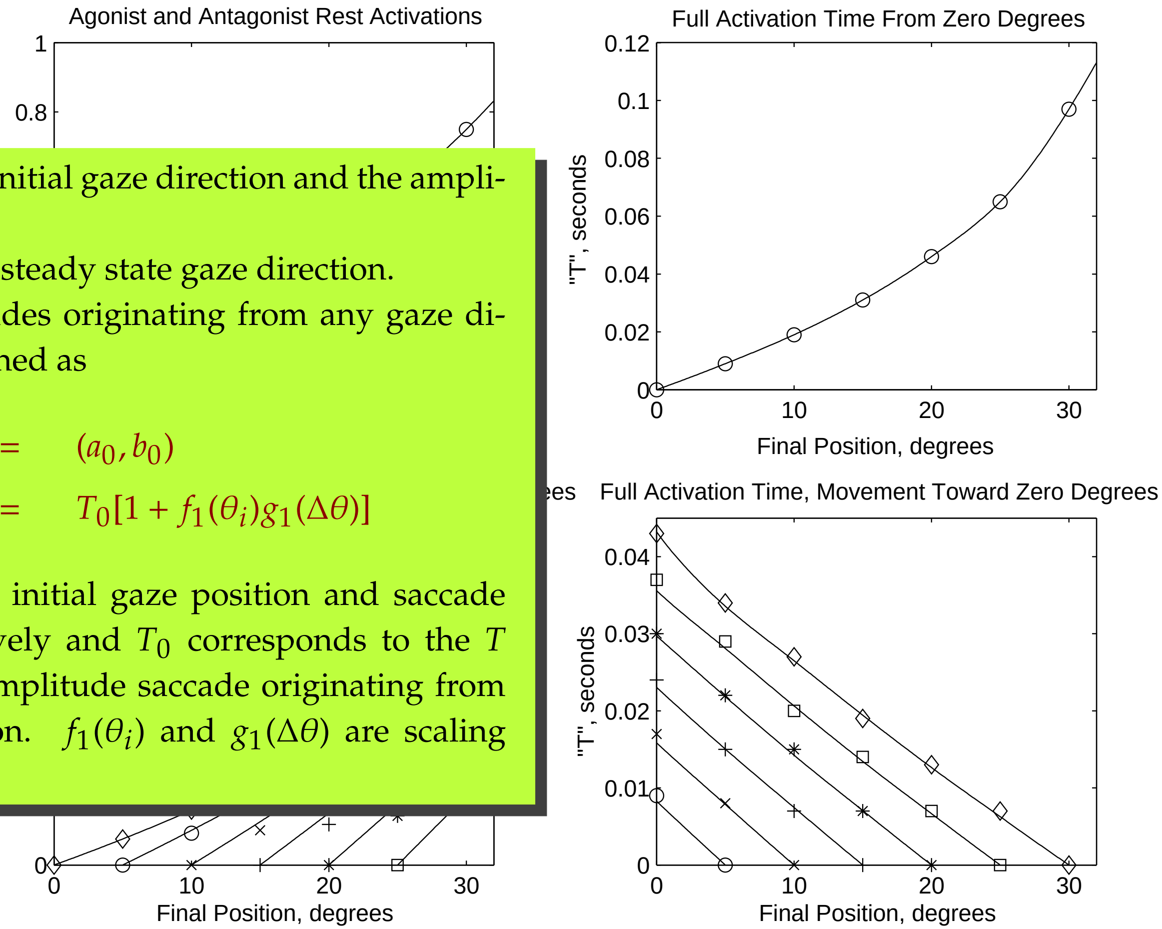


Figure 3: “Learning Curves”:Cubic Hermite interpolant splines developed from horizontal saccadic eye movements originating from the primary position. The bottom two figures demonstrate how the ‘T’ value changes with the initial gaze position.

Learning Curves



T : depends on the initial gaze direction and the amplitude of the saccade
a, b : depend on the steady state gaze direction.
(a_1, b_1, T_1) for saccades originating from any gaze direction can be obtained as

$$
\begin{aligned}
(a_1, b_1) &= (a_0, b_0) \\
T_1 &= T_0[1 + f_1(\theta_i) g_1(\Delta\theta)]
\end{aligned}
$$

θ_i and $\Delta\theta$ are the initial gaze position and saccade amplitude respectively and T_0 corresponds to the T value for a equal amplitude saccade originating from the primary position. $f_1(\theta_i)$ and $g_1(\Delta\theta)$ are scaling factors.

Figure 4: "Learning Curves":Cubic Hermite interpolant splines developed from horizontal saccadic eye movements originating from the primary position. The bottom two figures demonstrate how the 'T' value changes with the initial gaze position.

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- Planer eye movements ✓
- **Three-dimensional eye movements : Geometry**
- Eye as a simple mechanical control system
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Geometry of Eye Movements

- SO(3), the space of 3×3 rotation matrices, is the obvious choice for the configuration space.

$$
\begin{aligned}
\mathrm{SO}(3) &= \{\boldsymbol{R} : \mathbb{R}^3 \to \mathbb{R}^3 \mid (\boldsymbol{R}x, \boldsymbol{R}y)_{\mathbb{R}^3} = (x, y)_{\mathbb{R}^3}\,,\ \det\boldsymbol{R} = 1\} \\
&= \{\boldsymbol{R} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \boldsymbol{R}\boldsymbol{R}^T = \mathrm{Id},\ \det\boldsymbol{R} = 1\}
\end{aligned}
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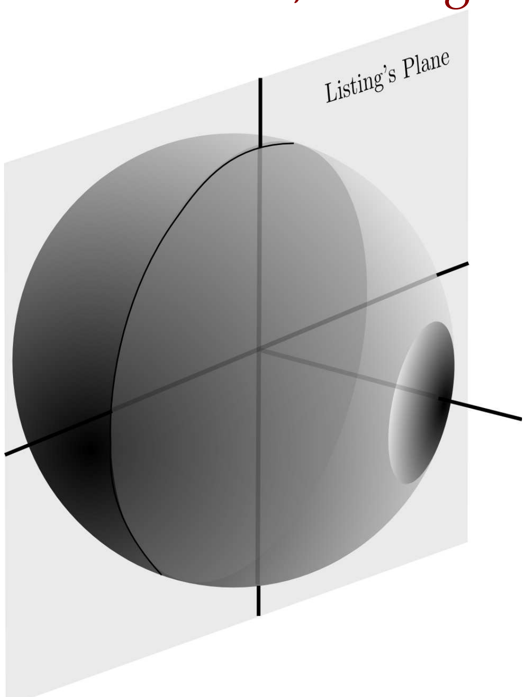

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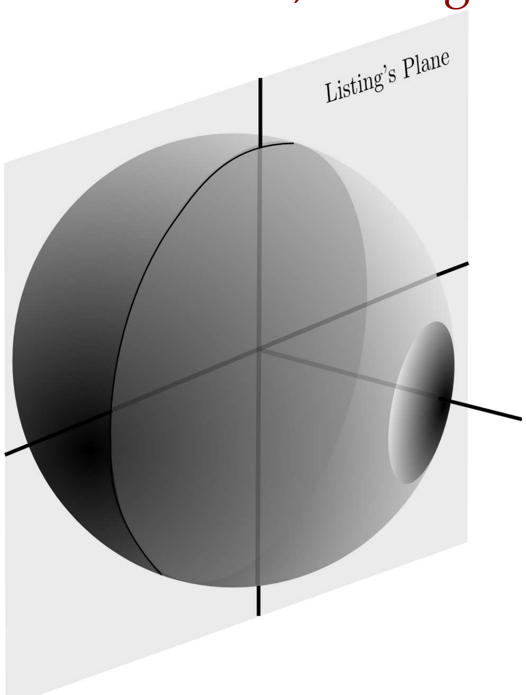

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Eye as a ***mechanical system with holonomic constraints***. Configuration space becomes a two dimensional submanifold of SO(3).
$\Rightarrow$ **"Listing Space (*List*)"**

Quaternions to represent rotations

- Space of quaternions are denoted by $\mathbf{Q}$.
- $a \in \mathbf{Q}$ can be written as $a_0 \overrightarrow{\mathbf{1}} + a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$.
- $\mathbf{vec}(a) = a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$
- $\mathbf{scal}(a) = a_0 \overrightarrow{\mathbf{1}}$
- The vector $a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$ will be identified with $(a_1, a_2, a_3) \in \mathbb{R}^3$ without any explicit mention of it.
- Quaternion product: $p.q = p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}$.

Thus we have maps,

$$\mathbf{vec} : \mathbf{Q} \to \mathbb{R}^3, \;\; a \mapsto (a_1, a_2, a_3),$$

and

$$\mathbf{scal} : \mathbf{Q} \to \mathbb{R}, \;\; a \mapsto a_0.$$

Quaternions to represent rotations

Each $q \in S^3$ (space of unit quaternions) can be written as

$$q = cos(\alpha/2)\overrightarrow{\mathbf{1}} + sin(\alpha/2)n_1\overrightarrow{\mathbf{i}} + sin(\alpha/2)n_2\overrightarrow{\mathbf{j}} + sin(\alpha/2)n_3\overrightarrow{\mathbf{k}}$$

where, $\alpha \in [0, \pi]$ and (n_1, n_2, n_3) is a unit vector in $\mathbb{R}^3$.

Define

$$\mathbf{rot} : \mathrm{S}^3 \to \mathrm{SO}(3)$$

as the standard map from S^3 into SO(3) which maps $cos(\alpha/2)\overrightarrow{\mathbf{1}} + sin(\alpha/2)n_1\overrightarrow{\mathbf{i}} + sin(\alpha/2)n_2\overrightarrow{\mathbf{j}} + sin(\alpha/2)n_3\overrightarrow{\mathbf{k}}$ to a rotation around the axis n by a counterclockwise angle α.

Quaternions to represent rotations

There are two explicit ways of describing this map. First,

$$\mathbf{rot}(q)(v_1, v_2, v_3) = \mathbf{vec}(q.(v_1 \overrightarrow{\mathbf{i}} + v_2 \overrightarrow{\mathbf{j}} + v_3 \overrightarrow{\mathbf{k}}).q^{-1}) \, .$$

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Second,

$$\mathbf{rot}(q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix} \in \mathsf{SO}(3).$$

Outline of the talk

- Anatomy of the eye ✓
- Planer eye movements ✓
- **Three-dimensional eye movements : Geometry**
- Eye as a simple mechanical control system
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  - Riemannian metric on ***List***.
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Local coordinates on ***List***

Let x_3 axis is aligned with the normal gaze direction,
then **Listing's law** amounts to a statement that all eye rotations have quaternion representations $q \in S^3$ with $q_3 = 0$.

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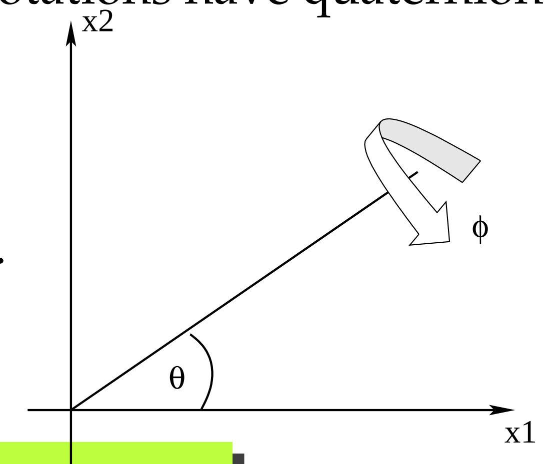

List is diffeomorphic to $\mathbb{P}^2$ (antipodal points identified).

Axis angle local coordinate system on *List*:
(θ, ϕ) describe the polar coordinate angle of the axis of rotation in the (x_1, x_2) plane and the angle of rotation around the axis respectively. Here we take $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$.

(Note: this fails when $\phi = 0$ or $\phi = 2\pi$ since in both cases the the corresponding rotation is identity regardless of the value of θ)

Riemannian metric on ***List***

Let's calculate the Riemannian metric on ***List*** induced from SO(3).

$$\begin{aligned} \mathrm{SO}(3) &= \{\boldsymbol{R} : \mathbb{R}^3 \to \mathbb{R}^3 \mid (\boldsymbol{R}x, \boldsymbol{R}y)_{\mathbb{R}^3} = (x, y)_{\mathbb{R}^3}\,,\ \det\boldsymbol{R} = 1\} \\ &= \{\boldsymbol{R} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \boldsymbol{R}\boldsymbol{R}^T = \mathrm{Id},\ \det\boldsymbol{R} = 1\} \end{aligned}$$

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$$\boldsymbol{\Omega}(t) = \boldsymbol{R}^T(t)\dot{\boldsymbol{R}}(t).$$

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The body ***angular velocity*** is defined as

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Since

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{R}(t)\boldsymbol{\Omega}(t), \qquad \boldsymbol{\Omega}^T(t) = -\boldsymbol{\Omega}(t),$$

the tangent space

$$\mathrm{T}_{\boldsymbol{R}}\mathrm{SO}(3) = \{\boldsymbol{R}\boldsymbol{\Omega} \mid \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}\}, \qquad \boldsymbol{R} \in \mathrm{SO}(3).$$

Then the tangent space to SO(3) at the identity:

$$\mathrm{T}_{\mathrm{Id}}\mathrm{SO}(3) = \{\boldsymbol{\Omega} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}\} = so(3).$$

Note that the space $so(3)$ is the Lie algebra of the Lie group SO(3).

Riemannian metric on *List*, (cont'd.)

Assuming that the eye as a perfect sphere, and its moment of inertia as $I_{3\times 3}$, the left invariant Riemannian metric on SO(3) given by,

$$\left\langle \mathbf{\Omega}(e_i), \mathbf{\Omega}(e_j) \right\rangle_I = \delta_{i,j} \, ,$$

where,

$$\mathbf{\Omega}(e_k) = \begin{bmatrix} 0 & \delta_{3,k} & -\delta_{2,k} \\ -\delta_{3,k} & 0 & \delta_{1,k} \\ \delta_{2,k} & -\delta_{1,k} & 0 \end{bmatrix},$$

and $\{\delta_{l,m}\}$ denotes the Kronecker delta function.

Riemannian metric on *List*, (cont'd.)

Now $\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{k}}$ is an orthonormal basis of $\mathsf{T}_{\overrightarrow{\mathbf{1}}} S^3$, and recall that $\mathbf{rot} : \mathrm{S}^3 \longrightarrow \mathsf{SO}(3)$, then

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ \sin(t/2) \\ 0 \\ 0 \end{bmatrix}\right) = \mathrm{e}^{t\boldsymbol{\Omega}(\mathrm{e}_1)}, \quad \mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ \sin(t/2) \\ 0 \end{bmatrix}\right) = \mathrm{e}^{t\boldsymbol{\Omega}(\mathrm{e}_2)}, \quad \mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ 0 \\ \sin(t/2) \end{bmatrix}\right) = \mathrm{e}^{t\boldsymbol{\Omega}(\mathrm{e}_3)}.$$

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Notice that,

$$\left.\frac{d}{dt}\right|_{t=0} \begin{bmatrix} \cos(t/2) \\ \sin(t/2) \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\overrightarrow{\mathbf{i}}}{2}, \qquad \left.\frac{d}{dt}\right|_{t=0} e^{t\boldsymbol{\Omega}(e_1)} = \boldsymbol{\Omega}(e_1).$$

Riemannian metric on *List*, (cont'd.)

Now $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$ is an orthonormal basis of $\mathsf{T}_{\vec{\mathbf{1}}}S^3$, and recall that $\mathbf{rot}: \mathrm{S}^3 \to \mathsf{SO}(3)$, then

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$$\begin{array}{ccc} S^3 & \xrightarrow{\mathbf{rot}} & \mathsf{SO}(3) \\ \downarrow & & \downarrow \\ \mathsf{T}_q S^3 & \xrightarrow{\mathbf{rot}_*} & \mathsf{T}_{\mathbf{rot}(q)}\mathsf{SO}(3) \end{array}$$

Therefore,

$$\mathbf{rot}_{*\vec{\mathbf{1}}}(\vec{\mathbf{i}}/2) = \boldsymbol{\Omega}(e_1), \qquad \mathbf{rot}_{*\vec{\mathbf{1}}}(\vec{\mathbf{j}}/2) = \boldsymbol{\Omega}(e_2), \qquad \mathbf{rot}_{*\vec{\mathbf{1}}}(\vec{\mathbf{k}}/2) = \boldsymbol{\Omega}(e_3).$$

Hence $\{\mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{i}}/2,\ \mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{j}}/2, \mathbf{rot}_{*\vec{\mathbf{1}}}\vec{\mathbf{k}}/2\}$ is an orthonormal frame in $\mathsf{T}_{\mathrm{Id}}(\mathsf{SO}(3))$.

Riemannian metric on *List*, (cont'd.)

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Thus, $\{\mathbf{rot}_{*q}\overrightarrow{\mathbf{i}}/2, \mathbf{rot}_{*q}\overrightarrow{\mathbf{j}}/2, \mathbf{rot}_{*q}\overrightarrow{\mathbf{k}}/2\}$ is an orthonormal basis of $\mathsf{T}_{\mathbf{rot}(q)}\mathsf{SO}(3)$ for all $q \in S^3$,

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The Riemannian metric on ***List*** has the form

$$g = ds^2 = \sum_{ij=1}^{n} g_{ij}dx_i dx_j$$

where

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle, \qquad (x_1, x_2) = (\theta, \phi).$$

Riemannian metric on ***List***, (cont'd.)

Let $\rho : [0, \pi] \times [0, 2\pi] \to S^3$,

$$\rho(\theta, \phi) = \begin{bmatrix} \cos(\phi/2) \\ cos(\theta)sin(\phi/2) \\ sin(\theta)sin(\phi/2) \\ 0 \end{bmatrix}.$$

$$\begin{array}{ccc} \textbf{\textit{List}} & \xrightarrow{\rho} & S^3 \\ \downarrow & & \downarrow \\ \mathsf{T}_{(\theta,\phi)}\textbf{\textit{List}} & \xrightarrow{\rho_*} & \mathsf{T}_{\rho(\theta,\phi)}S^3 \end{array}$$

Riemannian metric on *List*, (cont'd.)

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Then the Jacobian

$$\mathcal{J}(\rho)(\theta,\phi) = \left(\rho_{*(\theta,\phi)}(\tfrac{\partial}{\partial\theta}) \quad \rho_{*(\theta,\phi)}(\tfrac{\partial}{\partial\phi})\right) = \begin{pmatrix} 0 & -\frac{1}{2}sin(\phi/2) \\ -sin(\theta)sin(\phi/2) & \frac{1}{2}cos(\theta)cos(\phi/2) \\ cos(\theta)sin(\phi/2) & \frac{1}{2}sin(\theta)cos(\phi/2) \\ 0 & 0 \end{pmatrix}$$

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Also notice that

$$\rho(\theta,\phi).\overrightarrow{\mathbf{i}} = \begin{bmatrix} -cos(\theta)sin(\phi/2) \\ cos(\phi/2) \\ 0 \\ -sin(\theta)sin(phi/2) \end{bmatrix}, \quad \rho(\theta,\phi).\overrightarrow{\mathbf{j}} = \begin{bmatrix} -sin(\theta)sin(\phi/2) \\ 0 \\ cos(\phi/2) \\ cos(\theta)sin(\phi/2) \end{bmatrix}, \quad \rho(\theta,\phi).\overrightarrow{\mathbf{k}} = \begin{bmatrix} 0 \\ sin(\theta)sin(\phi/2) \\ -cos(\theta)sin(\phi/2) \\ cos(\phi/2) \end{bmatrix}.$$

Riemannian metric on *List*, (cont'd.)

For $\theta = 0$, it is easily observed that,

$$\begin{aligned}
\rho_{*(0,\phi)}(\frac{\partial}{\partial\theta}) &= sin(\phi/2)cos(\phi/2)\rho(0,\phi).\overrightarrow{\mathbf{j}} - sin^2(\phi/2)\rho(0,\phi).\overrightarrow{\mathbf{k}}, \\
\rho_{*(0,\phi)}(\frac{\partial}{\partial\phi}) &= \frac{1}{2}\rho(0,\phi).\overrightarrow{\mathbf{i}}.
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Therefore

$$\begin{aligned}
g_{11} &= \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = 4sin^2(\phi/2), \\
g_{12} &= \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} \right\rangle = 0, \\
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\end{aligned}$$

Thus, the Riemannian metric on ***List***

$$g = 4sin^2(\phi/2)d\theta^2 + d\phi^2.$$

Levi-Civita connection on *List*

Riemannian connection, $\nabla : \mathfrak{X}(\mathsf{M}) \times \mathfrak{X}(\mathsf{M}) \to \mathfrak{X}(\mathsf{M})$ of a Riemannian manifold M, is uniquely defined by the *Koszul* formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle \quad &= \quad \mathscr{L}_X\langle Y, Z\rangle + \mathscr{L}_Y\langle X, Z\rangle - \mathscr{L}_Z\langle X, Y\rangle \\ &\qquad - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle \end{aligned}$$

A Riemannian connection ∇ has the following properties:

$$\begin{aligned} \nabla_{fX+gY} &= f\nabla_X + g\nabla_Y, \\ \nabla_X(aY + bZ) &= a\nabla_X Y + b\nabla_X Z, \\ \nabla_X fY &= \mathscr{L}_X f Y + f\nabla_X Y, \end{aligned}$$

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y], \\ \mathscr{L}_X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(\mathsf{M})$, $f, g \in \mathfrak{F}(\mathsf{M})$ and $a, b \in \mathbb{R}$.

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for $X, Y, Z \in \mathfrak{X}(\mathsf{M})$, $f, g \in \mathfrak{F}(\mathsf{M})$ and $a, b \in \mathbb{R}$.

Using the subscripted coordinates (y_1, y_2) to denote (θ, ϕ) and *Christoffel symbols* Γ_{ij}^k

$$\nabla_{\partial y_i / \partial y_j} = \Gamma_{ij}^k \partial / \partial y_k,$$

Christoffel symbols are given by

$$\Gamma_{ij}^k = \sum_{h=1}^{2} \frac{g^{ih}}{2} \left\{ \frac{\partial g_{hj}}{\partial y_k} + \frac{\partial g_{hk}}{\partial y_j} - \frac{\partial g_{jk}}{\partial y_h} \right\} \quad i, j, k = 1, 2$$

Levi-Civita connection on *List* (cont'd.)

Now

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 4\sin^2(\phi/2) & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{(lower } g\text{-}ij\text{'s)}$$

and,

$$(g^{ij}) = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{4\sin^2(\phi/2)} & 0 \\ 0 & 1 \end{pmatrix}. \qquad \text{(upper } g\text{-}ij\text{'s)}$$

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Thus, we obtain expressions for Christoffel symbols,

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = -sin(\phi),$$

$$\Gamma^1_{12} = \frac{1}{2tan(\phi/2)}, \qquad \Gamma^1_{21} = \frac{1}{2tan(\phi/2)},$$

$$\Gamma^2_{12} = 0, \qquad \Gamma^2_{21} = 0,$$

$$\Gamma^1_{22} = 0, \qquad \Gamma^2_{22} = 0.$$

Geodesics on *List*

A geodesic is a curve whose length is the shortest distance between two points. Christoffel symbols can be used to compute geodesics. Let $\sigma(t) = (\theta(t), \phi(t))$ be a geodesic on ***List***. Then

$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) = 0,$$

where

$$\dot{\sigma}(t) = \left(\dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi}\right)$$

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$$\nabla_{\dot\sigma(t)}\dot\sigma(t) = \quad \nabla_{\left(\dot\theta\frac{\partial}{\partial\theta} + \dot\phi\frac{\partial}{\partial\phi}\right)}\left(\dot\theta\frac{\partial}{\partial\theta} + \dot\phi\frac{\partial}{\partial\phi}\right)$$

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Now,

$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) = \quad \dot{\theta}\nabla_{\frac{\partial}{\partial\theta}}\left(\dot{\theta}\frac{\partial}{\partial\theta}\right) + \dot{\theta}\nabla_{\frac{\partial}{\partial\theta}}\left(\dot{\phi}\frac{\partial}{\partial\phi}\right) + \dot{\phi}\nabla_{\frac{\partial}{\partial\phi}}\left(\dot{\theta}\frac{\partial}{\partial\theta}\right) + \dot{\phi}\nabla_{\frac{\partial}{\partial\phi}}\left(\dot{\phi}\frac{\partial}{\partial\phi}\right)$$

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Now,

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$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) = \ddot{\theta}\frac{\partial}{\partial\theta} + \ddot{\phi}\frac{\partial}{\partial\phi} + \dot{\theta}^2\boxed{\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta}} + \dot{\theta}\dot{\phi}\left(\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} + \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta}\right) + \dot{\phi}^2\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = 0$$

$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = \sum_{k=1}^{2}\Gamma_{11}^{k}\frac{\partial}{\partial y_k} = -\sin(\phi)\frac{\partial}{\partial\phi}$$

where $(y_1, y_2) = (\theta, \phi)$

Geodesics on *List*

A geodesic is a curve whose length is the shortest distance between two points. Christoffel symbols can be used to compute geodesics. Let $\sigma(t) = (\theta(t), \phi(t))$ be a geodesic on ***List***. Then

$$\nabla_{\dot{\sigma}(t)}\dot{\sigma}(t) = 0,$$

where

$$\dot{\sigma}(t) = \left(\dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi}\right)$$

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$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \sum_{k=1}^{2}\Gamma_{12}^{k}\frac{\partial}{\partial y_k} = \frac{1}{2\tan(\phi/2)}\frac{\partial}{\partial\theta}$$

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$$\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} = \sum_{k=1}^{2}\Gamma_{21}^{k}\frac{\partial}{\partial y_k} = \frac{1}{2\tan(\phi/2)}\frac{\partial}{\partial\theta}$$

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$$\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = \sum_{k=1}^{2}\Gamma_{22}^{k}\frac{\partial}{\partial y_k} = 0$$

where $(y_1, y_2) = (\theta, \phi)$

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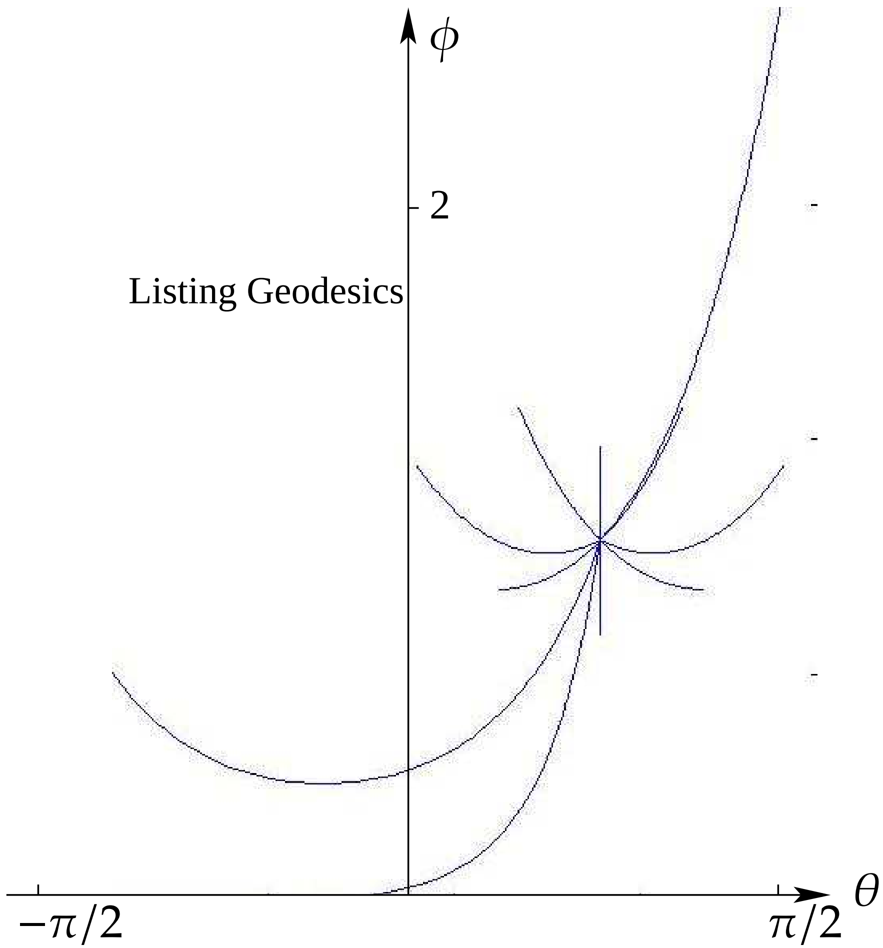
Therefore, the equations of geodesics

$$\ddot{\theta} + \frac{1}{tan(\phi/2)}\dot{\theta}\dot{\phi} = 0,$$
$$\ddot{\phi} - sin\phi\dot{\theta}^2 = 0.$$

Geodesics on *List*

Geodesics emanating from $(\pi/4, \pi/4)$



Curvature on *List*

The **curvature** $\mathcal{R}$ of a Riemannian manifold (M, g) is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(\mathsf{M})$ a mapping $\mathcal{R}(X, Y) : \mathfrak{X}(\mathsf{M}) \to \mathfrak{X}(\mathsf{M})$ given by

$$\mathcal{R}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathfrak{X}(\mathsf{M}),$$

where ∇ is the Levi-Civita connection of M.

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From the Christoffel symbols for the basis $\{\partial_\theta, \partial_\phi\}$, $\mathcal{R}$,

$$\mathcal{R}(\partial_\theta, \partial_\phi)\partial_\theta = \nabla_{\partial_\theta} \nabla_{\partial_\phi} \partial_\theta - \nabla_{\partial_\phi} \nabla_{\partial_\theta} \partial_\theta, \qquad \text{since } [\partial_\theta, \partial_\phi] = 0, (\text{Note: } \partial_\theta = \frac{\partial}{\partial \theta}).$$

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This evaluates to,

$$\begin{aligned} \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\theta &= -cos(\phi/2)\partial_\theta \\ \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi &= \frac{1}{4}\partial_\theta. \end{aligned}$$

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In particular, the Gauss curvature is given by,

$$\begin{aligned} K(\theta,\phi) &= \left\langle \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi, \partial_\theta \right\rangle / \langle \partial_\theta, \partial_\theta \rangle \\ &= 1/4 \end{aligned}$$

Outline of the talk

- Anatomy of the eye ✓
- Planer eye movements ✓
- Three-dimensional eye movements : Geometry ✓
- **Eye as a simple mechanical control system**
- Optimal control of the eye
- Conclusions and future directions

Eye as a simple mechanical control system

A “simple mechanical control system” (see Smale, 1970) consists the following:

- a ***configuration manifold*** Q,
- Riemannian metric g on Q that defines the kinetic energy function on the tangent bundle of Q,
- external forces as functions on the tangent bundle,
- any constraints on the system,
- control forces on the system as covector fields on the configuration manifold.

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- any constraints on the system,
- control forces on the system as covector fields on the configuration manifold.

For the eye movement system, ***List*** is the configuration manifold.

$g = 4sin^2(\phi/2)d\theta^2 + d\phi^2.$ is the Riemannian metric on ***List***.

Equations of motion

Let the Lagrangian of the system be

$$
\begin{aligned}
L(\theta,\phi,\dot{\theta},\dot{\phi}) &= \text{Kinetic Energy} - \text{Potential Energy} \\
&= \frac{1}{2}\left\|\dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi}\right\|^2 - V(\theta,\phi) \\
&= \frac{1}{2}\left\langle\dot{\theta}\frac{\partial}{\partial\theta},\dot{\phi}\frac{\partial}{\partial\phi}\right\rangle - V(\theta,\phi)
\end{aligned}
$$

Equations of motion

Let the Lagrangian of the system be

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&= \frac{1}{2}\left\langle \dot{\theta}\frac{\partial}{\partial\theta}, \dot{\phi}\frac{\partial}{\partial\phi} \right\rangle - V(\theta, \phi)
\end{aligned}
$$

Recall that

$$
\begin{aligned}
g_{11} &= < \partial_\theta, \partial_\theta > = 4sin^2(\phi/2), \\
g_{12} &= < \partial_\theta, \partial_\phi > = 0, \\
g_{22} &= < \partial_\phi, \partial_\phi > = 1.
\end{aligned}
$$

Equations of motion

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$$\begin{aligned} L(\theta,\phi,\dot\theta,\dot\phi) &= \text{Kinetic Energy} - \text{Potential Energy} \\ &= \frac{1}{2}\left\|\dot\theta\frac{\partial}{\partial\theta} + \dot\phi\frac{\partial}{\partial\phi}\right\|^2 - V(\theta,\phi) \\ &= \frac{1}{2}\left\langle\dot\theta\frac{\partial}{\partial\theta},\dot\phi\frac{\partial}{\partial\phi}\right\rangle - V(\theta,\phi) \end{aligned}$$

$$L(\theta,\phi,\dot\theta,\dot\phi) = 2\dot\theta^2\sin^2(\phi/2) + \frac{1}{2}\dot\phi^2 - V(\theta,\phi)$$

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\end{aligned}
$$

$$
L(\theta,\phi,\dot\theta,\dot\phi) = 2\dot\theta^2\sin^2(\phi/2) + \frac{1}{2}\dot\phi^2 - V(\theta,\phi)
$$

Euler-Lagrange equations:

$$
\frac{d}{dt}\frac{\partial L}{\partial\dot q^i} - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1,\dots,n.
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Equations of motion

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Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot q^i} - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1,\ldots,n.$$

Therefore the equations of motion:

$$\begin{aligned} \ddot\theta + \dot\theta\dot\phi cot(\phi/2) + \frac{1}{4}csc^2(\phi/2)\frac{\partial}{\partial\theta}V &= \frac{1}{4}csc^2(\phi/2)\tau_\theta \\ \ddot\phi - \dot\theta^2\sin(\phi) + \frac{\partial}{\partial\phi}V &= \tau_\phi. \end{aligned}$$

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- Anatomy of the eye ✓
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Optimal control

Case I: Generalized torques, τ_θ, τ_ϕ

Let $V(\theta, \phi) = sin^2(\phi/2)$.

Equations of motion:

$$\ddot{\theta} + \dot{\theta}\dot{\phi}\cot(\phi/2) = \frac{1}{4}csc^2(\phi/2)\tau_\theta$$
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Let $[z_1, z_2, z_3, z_4]' = [\theta, \dot{\theta}, \phi, \dot{\phi}]'$, then

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_2 z_4 cot(z_3/2) \\ z_4 \\ z_2^2 sin(z_3) - \frac{1}{2}sin(z_3) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{4}csc^2(z_3/2) \\ 0 \\ 0 \end{bmatrix}\tau_\theta + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\tau_\phi$$

Optimal control

We wish to control the state $(\theta, \dot{\theta}, \phi, \dot{\phi})$ from $(\theta_0, 0, \phi_0, 0)$ to $(\theta_1, 0, \phi_1, 0)$ in T unit of time, while minimizing the control energy,

$$\int_0^T \left[(\tau_\theta(t))^2 + (\tau_\phi(t))^2 \right] \mathrm{d}t.$$

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$$\int_0^T \left[(\tau_\theta(t))^2 + (\tau_\phi(t))^2\right] \mathrm{d}t.$$

Lagrangian:

$$L = \frac{1}{2}\left((\tau_\theta(t))^2 + (\tau_\phi(t))^2\right),$$

and denote the costate by λ. Construct the Hamiltonian

$$\begin{aligned}\mathcal{H}(z, \lambda) &= \lambda.\dot{z} - L(z) \\ &= \lambda_1 z_2 - \lambda_2 z_2 z_4 \cot(z_3/2) + \lambda_3 z_4 + \lambda_4 z_2^2 \sin(z_3) - \frac{1}{2}\lambda_4 \sin(z_3) \\ &\quad \frac{\lambda_2}{4\sin^2(z_3/2)}\tau_\theta + c\lambda_4 \tau_\phi + \frac{1}{2}\left((\tau_\theta(t))^2 + (\tau_\phi(t))^2\right)\end{aligned}$$

Optimal control

Hamilton's principle:

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i},$$

where $p_i = \frac{\partial L}{\partial \dot{q}^i}$, $\qquad i = 1, \ldots, n$.

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Hamiltonian system:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_2 z_4 cot(z_3/2) + 1/4 sin^2(z_3/2)\tau_\theta^* \\ z_4 \\ z_2^2 sin(z_3) - \frac{1}{2} sin(z_3) + \tau_\phi^* \\ 0 \\ -\lambda_1 + \lambda_2 z_4 cot(z_3/2) - 2\lambda_4 z_2 sin(z_3) \\ -\frac{1}{2}\lambda_2 z_2 z_4 csc^2(z_3/2) - \lambda_4 z_2^2 cos z_3 + \frac{1}{2}\lambda_4 cos(z_3) + \frac{1}{2}\lambda_2 cot(z_3) csc^2(z_3)\tau_\theta^* \\ \lambda_2 z_2 cot(z_3/2) - \lambda_3 \end{bmatrix}$$

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According to Pontryagin Maximum Principle (PMP), we can obtain:

$$\begin{aligned} \tau_\theta &= -\frac{\lambda_2}{4\sin^2(z_3/2)}, \\ \tau_\phi &= -\lambda_4. \end{aligned}$$

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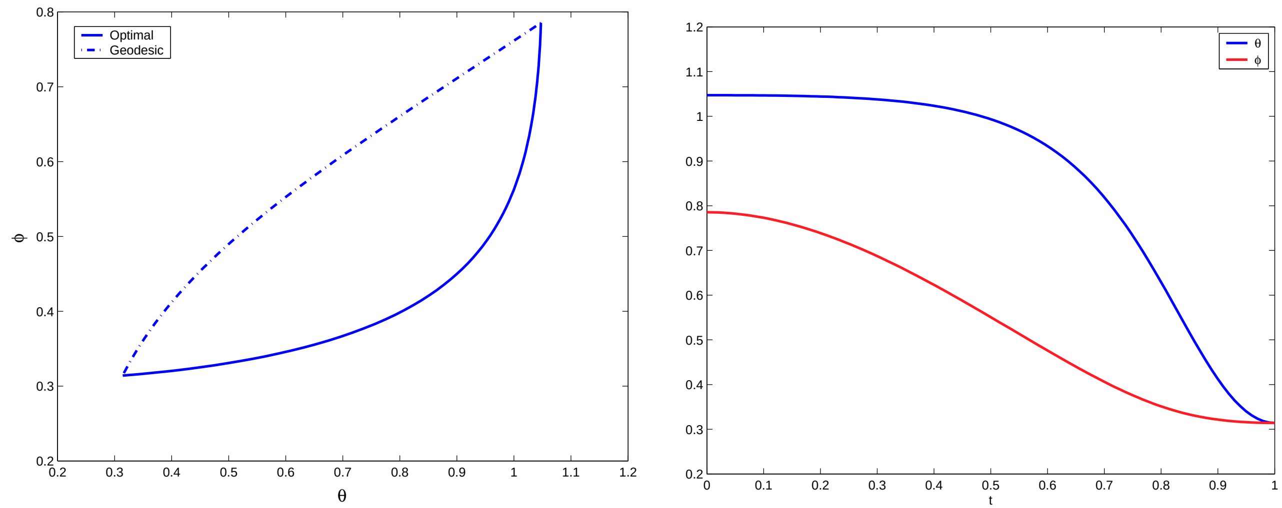
According to Pontryagin Maximum Principle (PMP), we can obtain:

$$\begin{aligned} \tau_\theta &= -\frac{\lambda_2}{4\sin^2(z_3/2)}, \\ \tau_\phi &= -\lambda_4. \end{aligned}$$

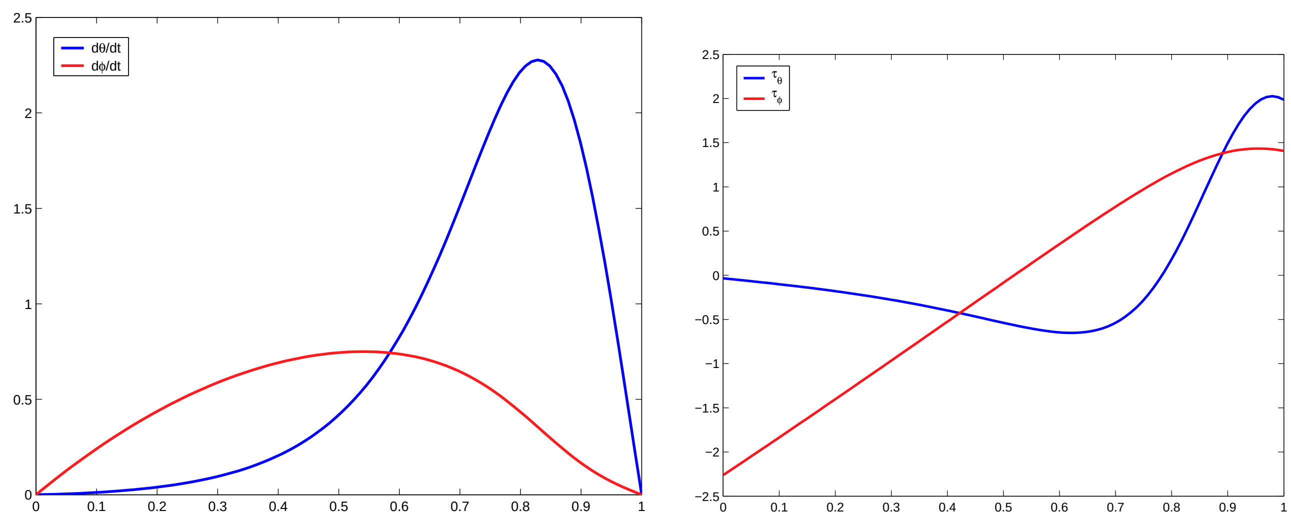
Thus the system becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_2 z_4 cot(z_3/2) - \frac{\lambda_2}{16} csc^4(z_3/2) \\ z_4 \\ z_2^2 sin(z_3) - \frac{1}{2} sin(z_3) - \lambda_4 \\ 0 \\ -\lambda_1 + \lambda_2 z_4 cot(z_3/2) - 2\lambda_4 z_2 sin(z_3) \\ (-\frac{1}{2}\lambda_2 z_2 z_4 csc^2(z_3/2) - \lambda_4 z_2^2 cos(z_3) + \\ \frac{1}{2}\lambda_4 cos(z_3/2) - \frac{\lambda_2^2}{16} csc^4(z_3/2) cot(z_3/2)) \\ \lambda_2 z_2 cot(z_3/2) - \lambda_3 \end{bmatrix}.$$

Optimal control



Optimal path from $(\pi/3, \pi/4)$ to $(\pi/10, \pi/10)$



$\dot{\theta}, \dot{\phi}$ and τ_θ, τ_ϕ

Optimal control

Case II: Simplified muscles

Each musculotendon consist of a linear spring with spring constant k_i, a damper with damping constant b_i, and an active force F_i.

Projecting the torques to *List*

$\theta \longrightarrow \theta + \delta\theta, \phi \longrightarrow \phi.$

Virtual work by the spring: $k_i(l_i - l_{i_0})\delta l = k_i(l_i - l_{i_0})\frac{\partial l_i}{\partial \theta} d\theta.$

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$$\tau_\theta = k_i(l_i - l_{i_0})\frac{\partial l_i}{\partial \theta}.$$

Also note, $\dot{l}_i = \dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi}.$

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Also note, $\dot{l}_i = \dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi}.$

Therefore for the damper: $F_{damp} = b_i\dot{l}_i = b_i(\dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi})$

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$\theta \longrightarrow \theta + \delta\theta, \phi \longrightarrow \phi$.

Virtual work by the spring: $k_i(l_i - l_{i_0})\delta l = k_i(l_i - l_{i_0})\frac{\partial l_i}{\partial \theta} d\theta$.

$$\tau_\theta = k_i(l_i - l_{i_0})\frac{\partial l_i}{\partial \theta}.$$

Also note, $\dot{l}_i = \dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi}$.

Therefore for the damper: $F_{damp} = b_i\dot{l}_i = b_i(\dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi})$

Then the torque with the active force F_i with $C_i = k_i(l_i - l_{i_0}) + b_i(\dot{\theta}\frac{\partial l_i}{\partial \theta} + \dot{\phi}\frac{\partial l_i}{\partial \phi})$:

$$\tau_\theta = \sum_{i=1}^{6} [F_i + C_i] \frac{\partial l_i}{\partial \theta} \qquad \tau_\phi = \sum_{i=1}^{6} [F_i + C_i] \frac{\partial l_i}{\partial \phi}$$

Optimal control

The optimal control problem becomes one of minimizing

$$\int_0^T \sum_{i=1}^{6} F_i^2 \mathrm{d}t.$$

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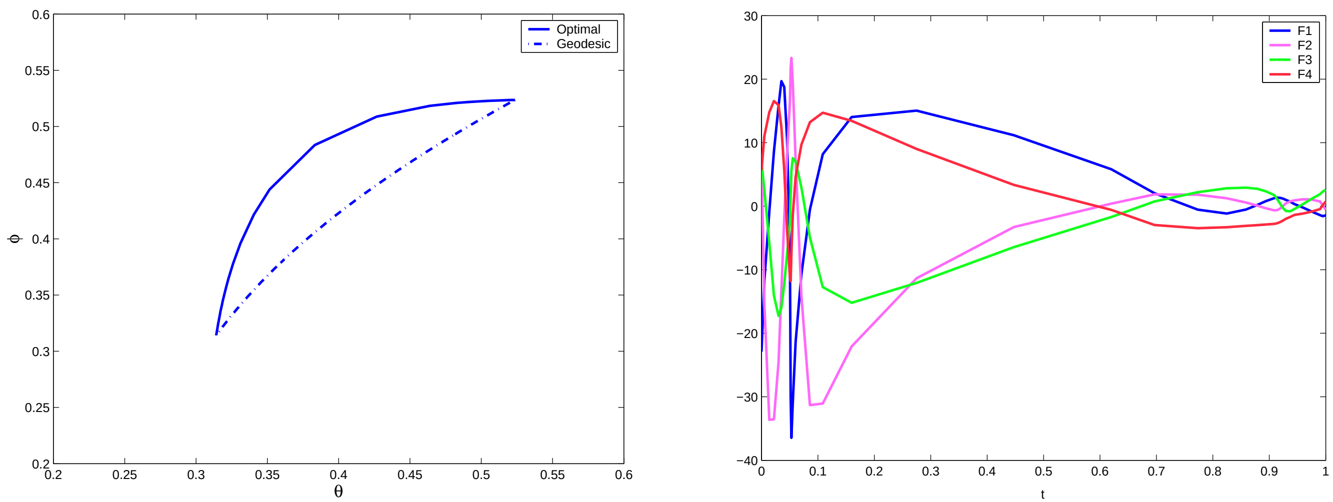
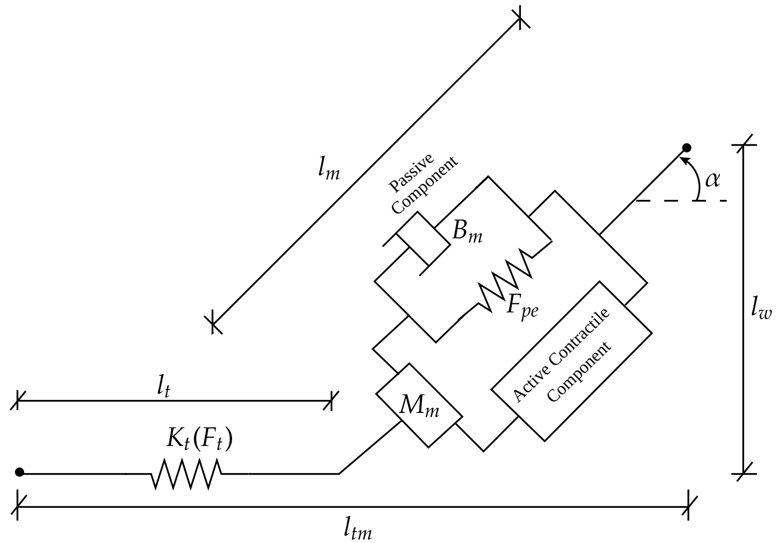


Optimal path and muscle forces, from $(\pi/6, \pi/6)$ to $(\pi/10, \pi/10)$

Optimal control

Case III: Hill-type muscles



Hill-type musculotendon

Optimal control

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$$F^i_{\text{total}} = F^i_t - (F^i_{act} + F^i_{pe} + B^i_m \dot{l}_i).$$

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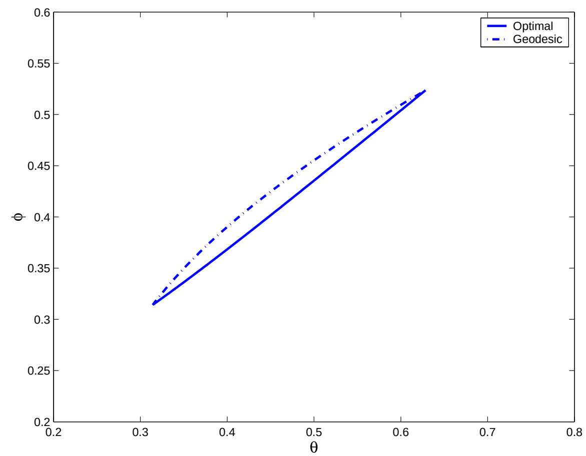
The problem beomes one of minimizing

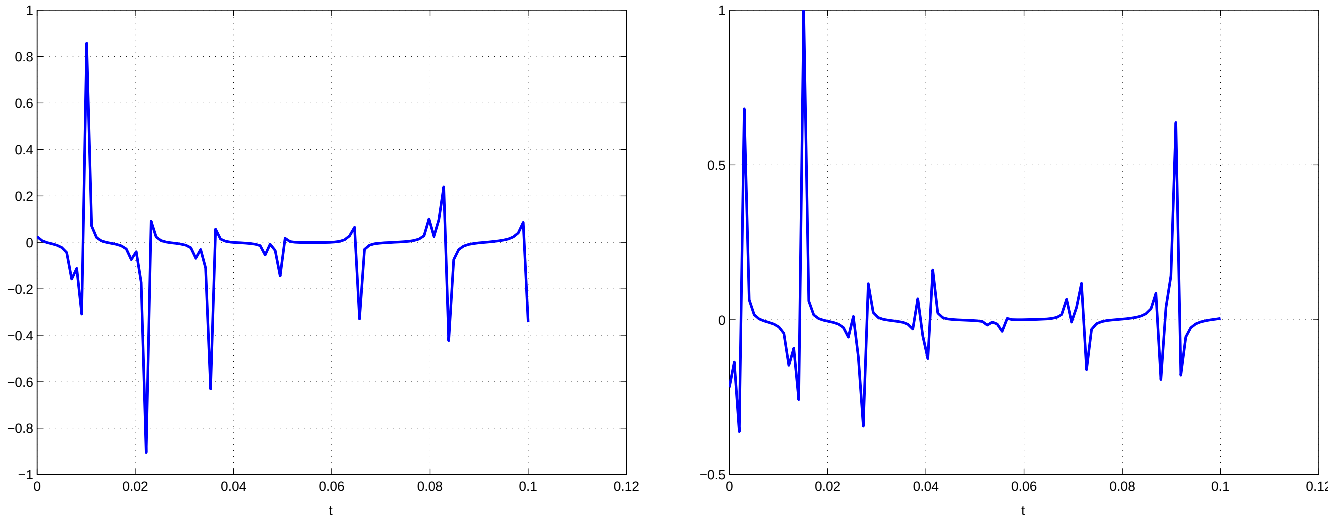
$$\int_0^T \sum_{i=1}^{6} \left[F^i_{act}(t)\right]^2 \mathrm{d}t.$$

Optimal control

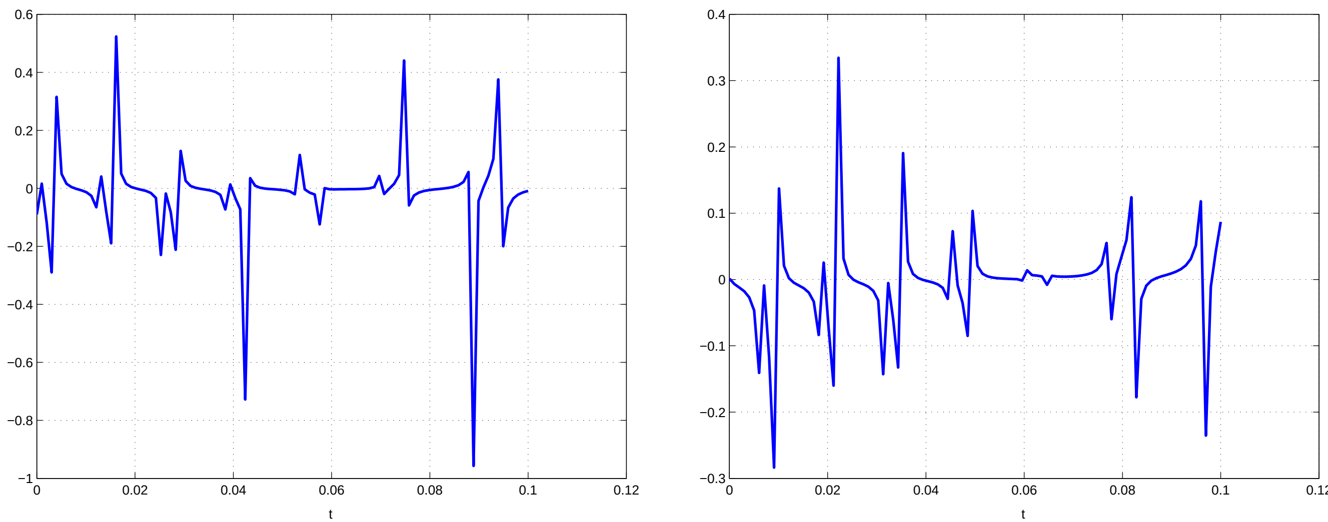


Optimal path from $(\pi/5, \pi/6)$ to $(\pi/10, \pi/10)$

Optimal control



Lateral and medial rectus muscle activities



Superior and inferior rectus muscle activities

Lengths of (Eye) Rotations

$$
\begin{aligned}
\ell(\sigma) &= \int_a^b \left\| \dot\theta \frac{\partial}{\partial\theta} + \dot\phi \frac{\partial}{\partial\phi} \right\| \, \mathrm{d}t \\
&= \int_a^b \sqrt{\dot\theta^2 g_{11} + 2\dot\theta\dot\phi g_{12} + \dot\phi^2 g_{22}} \, \mathrm{d}t \\
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From	**To**	**distance** (radians)		
(θ, ϕ)	(θ, ϕ)	$SO(3)$	**Geodesic on List**	**Min. energy on List**
$(\frac{\pi}{4}, \frac{\pi}{6})$	$(\frac{\pi}{8}, \frac{\pi}{8})$	0.219	0.222	0.324
$(\frac{\pi}{4}, \frac{\pi}{4})$	$(\frac{\pi}{8}, \frac{\pi}{6})$	0.359	0.368	0.368
$(\frac{\pi}{6}, \frac{\pi}{10})$	$(\frac{\pi}{8}, \frac{\pi}{4})$	0.476	0.480	0.482

Outline of the talk

- Anatomy of the eye ✓
- Planer eye movements ✓
- Three-dimensional eye movements : Geometry ✓
- Eye as a simple mechanical control system ✓
- Optimal control of the eye ✓
- **Conclusions and future directions**

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