

Worked Example 1

Solve the system

$$3x + 4y = 10 \quad (1)$$

$$2x + 5y = 9 \quad (2)$$

Method I: Rewrite one variable in terms of the other using one equation, and substitute in the other

We may use, for example, the second equation and rewrite y in terms of x .

$$\begin{aligned} 2x + 5y &= 9 \\ \text{Subtracting } 2x \text{ from both sides:} \quad 5y &= 9 - 2x \\ \text{Dividing both sides by 5:} \quad y &= \frac{9 - 2x}{5} \end{aligned} \quad (3)$$

Since we found this using the second equation, we should substitute this in the **other** equation (i.e. the equation (1))

$$\begin{aligned} 3x + 4y &= 10 \\ \text{Substitute for } y \text{ from (3):} \quad 3x + 4\left(\frac{9 - 2x}{5}\right) &= 10 \\ 3x + \frac{36 - 8x}{5} &= 10 \\ \text{Common denominator:} \quad \frac{15x}{5} + \frac{36 - 8x}{5} &= 10 \\ \frac{15x + 36 - 8x}{5} &= 10 \\ \frac{7x + 36}{5} &= 10 \\ \text{Multiplying both sides by 5:} \quad 7x + 36 &= 50 \\ \text{Subtracting 36 from both sides:} \quad 7x &= 14 \\ \text{Dividing both sides by 7:} \quad x &= 2 \end{aligned}$$

Now, using the expression for y , in terms of x (i.e. equation (3)) we can find the value of y ;

$$\begin{aligned} y &= \frac{9 - 2x}{5} \\ y &= \frac{9 - 2(2)}{5} \\ y &= \frac{9 - 4}{5} \\ y &= \frac{5}{5} \\ y &= 1 \end{aligned}$$

So we may write the solution to be $(2, 1)$.

This method may not be as elegant as the second method, but this is the only method that will work for nonlinear systems of equations. This method becomes very messy and tedious when we have more variables

Method II: Addition-subtraction method (Gaussian Elimination)

This method is the one used in computer programs which are used to solve systems of linear equations. This method can be easily extended to systems with many many variable. The disadvantage of this method, compared to the previous, is that, it cannot be used for systems of nonlinear equations.

It consists of two phases. (1) Elimination phase (2) Back substitution phase

Start with the Given system:

$$3x + 4y = 10 \quad (1)$$

$$2x + 5y = 9 \quad (2)$$

Elimination Phase

First multiply both sides of the FIRST equation by the coefficient of the x variable of the SECOND equation.

$$6x + 8y = 20 \quad (3)$$

Then multiply both sides of the SECOND equation by the coefficient of the x variable of the FIRST equation.

$$6x + 15y = 27 \quad (4)$$

Now subtract equation (3) equation FROM the equation (4)

$$\begin{array}{r} 6x + 15y = 27 \\ (-) \quad 6x + 8y = 20 \\ \hline + 7y = 7 \\ \hline \hline \end{array}$$

So we get $y = 1$, by dividing through by 7

At this point we have *eliminated* x from the equations.

Back substitution phase

We can substitute the value of y back in to either equation (1) or (2), and find the value of x .

If we use equation (1), we get,

$$3x + (4)(1) = 10$$

$$\text{Subtracting 4 from both sides:} \quad 3x = 6$$

$$\text{Dividing both sides by 3:} \quad x = 2$$

Here again, we get $x = 2$ and $y = 1$, so we may write $(2, 1)$ as the final answer.

Worked Example 2

An example with 3 variables.

Solve the system

$$3x + 2y + z = 10 \quad (1)$$

$$2x - 3y + 3z = 5 \quad (2)$$

$$5x + y + 2z = 13 \quad (3)$$

The first method becomes very messy for this example, so we will just use the Gaussian elimination (Method II) to solve this problem.

We can try to “Eliminate” any of the variables. Let’s eliminate x , first from equation (2), and then from equation (3), using equation (1):

Multiply both sides of equation (1) by the x coefficient of (2) i.e. 2

$$6x + 4y + 2z = 20 \quad (4)$$

Then multiply both sides of equation (2) by the x coefficient of (1) i.e. 3

$$6x - 9y + 9z = 15 \quad (5)$$

Subtract Equation (5) from Equation (4)

$$\begin{array}{r} 6x + 4y + 2z = 20 \\ (-) \quad 6x - 9y + 9z = 15 \\ \hline \qquad \qquad 13y - 7z = 5 \\ \hline \hline \end{array}$$

So, we get

$$13y - 7z = 5 \quad (6)$$

Now we have to start over with equation (3) and (1):

Multiply both sides of equation (1) by the x coefficient of (3) i.e. 5

$$15x + 10y + 5z = 50 \quad (7)$$

Then multiply both sides of equation (3) by the x coefficient of (1) i.e. 3

$$15x + 3y + 6z = 39 \quad (8)$$

Subtract Equation (8) from Equation (7)

$$\begin{array}{r} 15x + 10y + 5z = 50 \\ (-) \quad 15x + 3y + 6z = 39 \\ \hline \qquad \qquad 7y - z = 11 \\ \hline \hline \end{array}$$

So, we get

$$7y - z = 11 \quad (9)$$

Now note that equation (6) and equation (9) has no x term.

$$13y - 7z = 5 \tag{6}$$

$$7y - z = 11 \tag{9}$$

Now we can try to eliminate either y or z from these two equations. Eliminating z would be easier in this example. But I will try to eliminate y (there is no particular reason to do the harder one, but I will do it any way)

Multiply both sides of equation (6) by the y coefficient of equation (9) i.e. 7

$$117y - 49z = 35 \tag{10}$$

Now multiply both sides of equation (9) by the y coefficient of equation (6) i.e. 13

$$91y - 13z = 143 \tag{11}$$

Now subtract equation (11) equation FROM the equation (10)

$$\begin{array}{r} 117y - 49z = 35 \\ (-) \quad 91y - 13z = 143 \\ \hline \qquad \qquad \qquad -36z = -108 \\ \hline \hline \end{array}$$

So eventually we get, $z = (-108)/(-36)$, so $z = 3$.

This completes the “elimination” and we can start the back substitution phase.

Use equation (9),

$$7y - 3 = 11$$

Adding 3 from both sides: $7y = 14$

Dividing both sides by 7: $y = 2$

Finally, we may use equation (1) to find x :

$$3x + (2)(2) + 3 = 10$$

$$3x + 4 + 3 = 10$$

$$3x + 7 = 10$$

Subtracting 7 from both sides: $3x = 3$

Dividing both sides by 3: $x = 1$

So we have $x = 1$, $y = 2$, $z = 3$ as the solutions for x , y and z .

Worked Example 3: Application/word problem - System of linear equations

Q: Find the equation of the parabola which passes through the points $(0, 2)$, $(1, 4)$ and $(-1, 2)$.

A:

First recall that the equation of a parabola is of the form $y = ax^2 + bx + c$. So, all we have to do is to find a , b and c .

Substituting the point $(0, 2)$, i.e. setting $x = 0$ and $y = 2$, we get $2 = a(0)^2 + b(0) + c$. So, immediately we see that $c = 2$.

Substituting the point $(1, 4)$, i.e. setting $x = 1$ and $y = 4$, we get $4 = a(1)^2 + b(1) + c$. We can also substitute the value for c , because we know it. So we get, $4 = a + b + 2$, and subtracting 2 from both sides, we get

$$a + b = 2 \tag{1}$$

Substituting the point $(-1, 2)$, i.e. setting $x = -1$ and $y = 2$, we get $2 = a(-1)^2 + b(-1) + c$. We can also substitute the value for c , because we know it. So we get, $2 = a - b + 2$, and subtracting 2 from both sides, we get

$$a - b = 0 \tag{2}$$

Now we have a system of linear equations

$$a + b = 2 \tag{1}$$

$$a - b = 0 \tag{2}$$

This can be easily solved by adding equations (1) and (2) to get,

$$2a = 2 \tag{3}$$

So, we get, $a = 1$.

Then substituting for a in equation (1), we get, $b = 1$ as well.

Therefore, the required parabola is $y = x^2 + x + 2$.

You may (and probably should anyway) plug-in the given coordinate points and verify this is correct.

Worked Example 4: Application/word problem - System of linear equations

Q: Find the coordinates of the point at which the two straight lines $x + 2y - 3 = 0$ and $3x - 4y + 1 = 0$ intersect.

A:

First note that if the two given lines intersect, then the two lines should pass through some common point.

We write down the system of equations:

$$x + 2y - 3 = 0 \tag{1}$$

$$3x - 4y + 1 = 0 \tag{2}$$

If you want, we can rewrite these in the usual format: We write down the system of equations:

$$x + 2y = 3 \tag{1}$$

$$3x - 4y = -1 \tag{2}$$

We can either the Gaussian elimination or the substitution method to solve this system of equations.

We can write x in terms of y easily using equation (1).

$$x = 3 - 2y \quad (3)$$

Now, we can substitute this in equation (2),

$$3(3 - 2y) - 4y = -1$$

Simplifying this, we get, $9 - 6y - 4y = -1$,

combining the y terms, $9 - 10y = -1$,

then subtracting 9 from both sides, $-10y = -10$,

dividing both sides by -10 , we finally get $y = 1$.

So, substituting $y = 1$ in equation (3), we get $x = 1$.

So the coordinates of the point of intersection is $(1, 1)$

Note that, solving a system of linear equations can be thought of as finding the point of intersection of lines

Worked Example 5: Application/word problem - System of nonlinear equations

Q: Find the point(s) of intersection between the circle $x^2 + y^2 = 4$ and the straight line $x + y + 1 = 0$

A: We get a system of equations but it is not linear (i.e. it is not the intersection of lines anymore)

$$x^2 + y^2 = 4 \quad (1)$$

$$x + y + 1 = 0 \quad (2)$$

The “trick” in such problems is to write one variable in terms of the other using one equation and substitute it in to the other equation.

In this particular problem, the job is easy, because one equation is a line. So, we may use equation (2) to write

$$y = -x - 1. \quad (3)$$

Substituting this in equation (1), we get,

$$x^2 + (-x - 1)^2 = 4.$$

Which is $x^2 + (-x - 1)(-x - 1) = 4$.

Multiplying out the two parenthesized terms, we get, $x^2 + x^2 + 2x + 1 = 4$.

Subtracting 4 from both sides and simplifying this, we get, $2x^2 + 2x - 3 = 0$.

This is a quadratic equation, which we can solve using the quadratic formula. Noting that $a = 2$, $b = 2$, $c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, $x = \frac{-2 \pm \sqrt{4 - (4)(2)(-3)}}{(2)(2)}$, which gets simplifies to $x = \frac{-1 \pm \sqrt{7}}{2}$,

or simply using a calculator, $x = 0.822875656$ or $x = -1.822875656$

Using equation (3), we can the solution for y corresponding to each value of x .

Worked Example 6: Application/word problem - System of nonlinear equations

Q: Find the point(s) of intersection between the circle $x^2 + y^2 = 1$ and the parabola $y = x^2$

A: We get a system of equations

$$x^2 + y^2 = 1 \tag{4}$$

$$y = 2x^2 \tag{5}$$

We can substitute $y = x^2$, which we get from equation (2), back in equation (1).

Then we obtain $x^2 + 4x^4 = 1$

Set $z = x^2$, and we can convert this to a quadratic equation: $z + z^2 = 1$, rewriting in the standard form, this is $z^2 + 4z - 1 = 0$. We have to solve for z using the quadratic formula. $z = \frac{-4 \pm \sqrt{4 - (4)(1)(-1)}}{2}$.

$$\text{So, } z = \frac{-4 \pm \sqrt{8}}{2}$$

Recall that $z = x^2$, so that we can accept only the positive solutions for z .

Note that $z = \frac{-4 - \sqrt{8}}{2}$ is negative and $z = \frac{-4 + \sqrt{8}}{2}$ is positive.

$$\text{So, } x = \pm \sqrt{\frac{-4 + \sqrt{8}}{2}}$$

$$\text{Hence, } y = x^2 = \frac{-4 + \sqrt{8}}{2}$$

(You could simplify the solutions a bit noticing that $\sqrt{8} = 2\sqrt{2}$, so, $x = \pm\sqrt{-2 + \sqrt{2}}$ and $y = -2 + \sqrt{2}$)

So the points of intersection are, $(\sqrt{-2 + \sqrt{2}}, -2 + \sqrt{2})$ and $(-\sqrt{-2 + \sqrt{2}}, -2 + \sqrt{2})$

Exercise Problems Solve the following systems of equations:

1.
$$\begin{cases} x + y = 2 \\ 2x + y = 1 \end{cases}$$

4.
$$\begin{cases} 3x + 2y + z = 18 \\ 2x + y + z = 13 \\ x + 2y + z = 14 \end{cases}$$

7.
$$\begin{cases} xy = 1 \\ x^2 + y^2 = 1 \end{cases}$$

2.
$$\begin{cases} 3x + 2y = 6 \\ 12x + y = 3 \end{cases}$$

5.
$$\begin{cases} x^2 + 2x + y^2 = 0 \\ x - y = 0 \end{cases}$$

8.
$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y = 1 \\ x + z = 2 \end{cases}$$

3.
$$\begin{cases} x + y + z = 0 \\ 2x + y + z = -1 \\ x + y + 2z = 1 \end{cases}$$

6.
$$\begin{cases} xy = 2 \\ y = 2x + 1 \end{cases}$$