Partial Fractions (§12.7 & §12.8)

The following table tells you the form of partial fractions to pick. Here $p(x)$ is any polynomial with degree less than the degree of the denominator. (For example, if the denominator is degree 2, then $p(x)$ has to be linear polynomial or a constant term. If the degree of the denominator is 3 then, $p(x)$ has to be a quadratic polynomial, linear polynomial or a constant term.) **YOU HAVE TO KNOW THE FOLLOWING TABLE !**

Name	Rational Function Form	Partial Fraction form
Two Linear Factors	$\dfrac{p(x)}{(x+a)(x+b)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}$
Three Linear Factors	$\dfrac{p(x)}{(x+a)(x+b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$
Irreducible Quadratic	$\dfrac{p(x)}{(ax^2+bx+c)(x+d)}$; $b^2-4ac<0$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x+d)}$
Linear Factors with Multiplicity	$\dfrac{p(x)}{(x+a)^2(x+b)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+b)}$

If the denominator is not factored, you WILL HAVE TO factor it first, *unless it is irreducible.*

Then the problem is to find the constants A, B, C, $\ldots$.

Worked Examples

- $\dfrac{5x+4}{(x+2)(x-3)}$

  We have to use the form $\dfrac{5x+4}{(x+2)(x-3)}=\dfrac{A}{(x+2)}+\dfrac{B}{(x-3)}$.

  Take the right hand side to a common denominator, while leaving the left side as it is.

$$\frac{5x+4}{(x+2)(x-3)} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

  Expand the parenthesis of the right hand side expression. $\dfrac{5x+4}{(x+2)(x-3)}=\dfrac{Ax-3A+Bx+2B}{(x+2)(x-3)}$

  Group the terms. $\dfrac{5x+4}{(x+2)(x-3)}=\dfrac{(A+B)x-3A+2B}{(x+2)(x-3)}$

  Compare the coefficients of the numerator (WE WANT THEM TO BE THE SAME)

$$\begin{cases}\text{Coefficient of } x: & A+B=5\\ \text{Constant term}: & -3A+2B=4\end{cases}$$

  So, we have to solve this system of equations to find A and B. If we do it properly, we will get, $A=\frac{6}{5}$ and $B=\frac{19}{5}$.

  Hence the partial fraction expansion is:

$$\frac{5x+4}{(x+2)(x-3)} = \frac{(6/5)}{(x+2)}+\frac{(19/5)}{(x-3)}$$

- $\dfrac{2x+1}{(x^2+4)(x+5)}$

  We have to use the form $\dfrac{2x+1}{(x^2+4)(x+5)} = \dfrac{Ax+B}{(x^2+4)} + \dfrac{C}{(x+5)}$, since $b^2 - 4ac = 0^2 - (4)(1)(4) = -16 < 0$.

  Take the right hand side to a common denominator, while leaving the left side as it is.

$$\frac{2x+1}{(x^2+4)(x+5)} = \frac{(Ax+B)(x+5) + C(x^2+4)}{(x^2+4)(x+5)}$$

  Expand the parenthesis of the right hand side expression. $\dfrac{2x+1}{(x^2+4)(x+5)} = \dfrac{Ax^2+5Ax+Bx+5B+Cx^2+4C}{(x^2+4)(x+5)}$

  Group the terms. $\dfrac{2x+1}{(x^2+4)(x+5)} = \dfrac{(A+C)x^2+(5A+B)x+(5B+4C)}{(x^2+4)(x+5)}$

  Compare the coefficients of the numerator (WE WANT THEM TO BE THE SAME)

$$\begin{cases} \text{Coefficient of } x^2 : & A+C=0 \\ \text{Coefficient of } x : & 5A+B=2 \\ \text{Constant term} : & 5B+4C=1 \end{cases}$$

  So, we have to solve this system of equations to find A and B and C. If we do it properly, we will get, $A = \frac{9}{29}$, $B = \frac{13}{29}$ and $C = \frac{-9}{29}$.

  Hence the partial fraction expansion is:

$$\frac{2x+1}{(x^2+4)(x+5)} = \frac{\left(\left(\frac{9}{29}\right)x + \left(\frac{13}{29}\right)\right)}{(x^2+4)} + \frac{\left(\frac{-9}{29}\right)}{(x+5)}$$

- $\dfrac{4x+3}{(x+2)^2(x+1)}$

  We have to use the form $\dfrac{4x+3}{(x+2)^2(x+1)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+1)}$, since factor $(x+2)$ has multiplicity 2.

  Take the right hand side to a common denominator, while leaving the left side as it is.

$$\frac{4x+3}{(x+2)^2(x+1)} = \frac{A(x+2)(x+1) + B(x+1) + C(x+2)^2}{(x+2)^2(x+1)}$$

  Expand the parenthesis of the right hand side expression.

$$\begin{aligned} \frac{4x+3}{(x+2)^2(x+1)} &= \frac{A(x^2+3x+2) + B(x+1) + C(x^2+4x+4)}{(x+2)^2(x+1)} \\ &= \frac{Ax^2+3Ax+2A+Bx+B+Cx^2+4Cx+4C}{(x+2)^2(x+1)} \end{aligned}$$

  Group the terms. $\dfrac{4x+3}{(x+2)^2(x+1)} = \dfrac{(A+C)x^2+(3A+B+4C)x+(2A+B+4C)}{(x+2)^2(x+1)}$

Compare the coefficients of the numerator (WE WANT THEM TO BE THE SAME)

$$\begin{cases} \text{Coefficient of } x^2: & A + C = 0 \\ \text{Coefficient of } x: & 3A + B + 4C = 4 \\ \text{Constant term}: & 2A + B + 4C = 3 \end{cases}$$

So, we have to solve this system of equations to find A and B and C. If we do it properly, we will get, $A = 1$, $B = 5$ and $C = -1$.

Hence the partial fraction expansion is:

$$\frac{4x+3}{(x+2)^2(x+1)} = \frac{1}{(x+2)} + \frac{5}{(x+2)^2} + \frac{-1}{(x+1)}$$

NOTE: If the degree of the numerator is greater than the degree of the denominator, use polynomial (long or synthetic) division to find the quotient and residue.

- $\dfrac{x^3+6x^2+11x+7}{(x^2+3x+2)}$

Since the numerator is degree 3 and the denominator is degree 2, we use polynomial division to obtain the quotient and residue.
$\dfrac{x^3+6x^2+11x+7}{(x^2+3x+2)} = (x+3) + \dfrac{1}{(x^2+3x+2)}$
Note that the denominator can be factored $x^2 + 3x + 2 = (x+2)(x+3)$. So we may write,

$$\frac{x^3+6x^2+11x+7}{(x^2+3x+2)} = (x+3) + \frac{1}{(x+1)(x+2)}$$

We can find the partial fraction expansion only for $\dfrac{1}{(x+1)(x+2)}$ term:
$\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{(x+1)} - \dfrac{1}{(x+2)}$. *(fill in the details ...)*
So, we write:

$$\frac{x^3+6x^2+11x+7}{(x^2+3x+2)} = (x+3) + \frac{1}{(x+1)} - \frac{1}{(x+2)}$$

Exercises

Find the partial fraction expansions of the following rational functions

1. $\dfrac{2x+5}{(x+2)(x+3)}$
2. $\dfrac{x^2+5x+5}{(x+2)^2(x+1)}$
3. $\dfrac{x^2+4x+4}{(x+2)^2(x+1)}$
4. $\dfrac{x^2+x+1}{(x^2+1)(x+1)}$
5. $\dfrac{x+1}{(x^2+1)(x^2+2)}$
6. Exercise 12.7 (page 954) Problems 17, 18, 19, 20, 21, 22)