Roots of Polynomial Equations (§12.2,3,4,5)

Factor Theorem *(§12.3)*
Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, if for some number r, if we get $f(r) = 0$ then, we can say,

- $(x - r)$ is a factor the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$
- $x = r$ is a root the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$

Remainder Theorem *(§12.3)*
When a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ is divided by $(x - r)$, the remainder is $f(r)$.
(It is clear that, if $(x-r)$ is a factor of $f(x)$, then the remainder is zero. In other words, the 'Factor Theorem' is merely a corollary of the remainder theorem.)

Rational Roots Theorem *(§12.5)*
Consider the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$, where all the coefficients $a_0, \ldots, a_n$ are integers. If p/q is a rational number (i.e. p and q are integers and have NO COMMON FACTORS except ± 1) and if it is a root of this equation, then,

- p is a factor of *the constant term* (a_0) and,
- q is a factor of *the leading term* (a_n).

The Fundamental Theorem of Algebra *(§12.4) - Special case for real roots*
A polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$, of degree n, can have *AT MOST* n real roots.

We can use the "Rational Roots Theorem" and the "Factor Theorem" along with the polynomial division (long or synthetic) to find the rational roots of a polynomial equation.

Examples:

1. Solve $2x^3 - 9x^2 + 10x - 3 = 0$

   Factors of the constant term (-3): $(-3) \times 1$, $3 \times (-1)$; So, p (as in the 'Rational Roots Theorem') can be ± 3 or ± 1

   Factors of the leading term (2): 2×1, $(-2) \times (-1)$; So, q (as in the 'Rational Roots Theorem') can be ± 2 or ± 1

   So, the "possible" rational roots are $\frac{p}{q}$: $\pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{1}$

   We can test if each of these (8 possibilities) are roots of the given equation, or, just find one root out of the possible roots and divide by the corresponding factor to obtain a quadratic polynomial and solve the quadratic polynomial using the quadratic formula.

x	1	-1	3	-3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$
$f(x)$	0	-24	0	-168	0	10.5	-1.5	-45

   We immediately see the three roots of this polynomial equation to be $x = 1, 3, \frac{1}{2}$.

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2. Solve $4x^3 - 12x^2 + 7x + 2 = 0$

   Factors of the constant term 2: 2×1, $(-2) \times (-1)$; So, p can be ± 2 or ± 1

   Factors of the leading term 4: 4×1, $(-4) \times (-1)$, 2×2, $(-2) \times (-2)$; So, q can be ± 4, ± 2 or ± 1

   So, the "possible" rational roots are $\frac{p}{q}$: $\pm\frac{4}{1}, \pm\frac{2}{1}, \pm\frac{1}{1}, \pm\frac{4}{2}, \pm\frac{2}{2}, \pm\frac{1}{2}$

   Since $\pm\frac{2}{2} = \pm\frac{1}{1} = \pm 1$, and $\pm\frac{4}{2} = \pm\frac{2}{1} = \pm 2$ we have 8 possibilities: $\pm 4, \pm 2, \pm 1, \pm 1/2$.

   We can test if each of these (8 possibilities) are roots of the given equation. According to the "Rational Roots Theorem" these are the ONLY POSSIBLE rational roots for this equation.

x	4	-4	2	-2	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$
$f(x)$	94	-474	0	-92	1	21	3	-5

   We see that the ONLY rational root is $x = 2$.

   The "Fundamental Theorem of Algebra" says that we might have two more real solutions. We may use long division or synthetic division divide the given polynomial by the factor we just found, $(x - 2)$.

Long Division

$$
\begin{array}{r|l}
 & 4x^2 - 4x - 1 \\
\hline
x - 2) & 4x^3 - 12x^2 + 7x + 2 \\
 & -4x^3 + 8x^2 \\
\hline
 & \quad -4x^2 + 7x \\
 & \quad 4x^2 - 8x \\
\hline
 & \qquad -x + 2 \\
 & \qquad x - 2 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrr}
 & 4 & -12 & 7 & 2 \\
2 & & 8 & -8 & -2 \\
\hline
 & 4 & -4 & -1 & 0
\end{array}
$$

Either way, we see that $4x^3 - 12x^2 + 7x + 2 = (x-2)(4x^2 - 4x - 1)$.

All we have to do, in order to complete the problem, is to solve $4x^2 - 4x - 1 = 0$, which can be easily done using the quadratic formula (note that it is impossible to factor this using rational numbers!).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - (4)(4)(-1)}}{(2)(4)} = \frac{4 \pm \sqrt{16+16}}{(2)(4)} = \frac{4 \pm \sqrt{32}}{(2)(4)} = \frac{1 \pm \sqrt{2}}{2}.$$

So, the roots of $4x^3 - 12x^2 + 7x + 2 = 0$ are, $x = 2, \dfrac{1+\sqrt{2}}{2}, \dfrac{1-\sqrt{2}}{2}$.

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3. Solve $x^3 - 4x^2 + 6x - 4 = 0$

Factors of the constant term -4: $(-4) \times 1$, $4 \times (-1)$, $(-2) \times 2$, $2 \times (-2)$; So, p can be ± 4, ± 2 or ± 1

Factors of the leading term 1: 1×1, $(-1) \times (-1)$; So, q can be ± 1

So, the "possible" rational roots are $\dfrac{p}{q}$: $\pm\dfrac{4}{1}, \pm\dfrac{2}{1}, \pm\dfrac{1}{1}$.

We can test if each of these (6 possibilities) are roots of the given equation.

x	4	-4	2	-2	1	-1
$f(x)$	20	-156	0	-40	-1	-15

We see that the ONLY rational root is $x = 2$.

The "Fundamental Theorem of Algebra" says that we might have two more real solutions. We may use long division or synthetic division divide the given polynomial by the factor we just found, $(x-2)$.

Long Division

$$
\begin{array}{rl}
 & \;\; x^2 - 2x + 2 \\
\cline{2-2}
x - 2 & \big)\; x^3 - 4x^2 + 6x - 4 \\
 & \;\; -x^3 + 2x^2 \\
\cline{2-2}
 & \qquad -2x^2 + 6x \\
 & \qquad\;\; 2x^2 - 4x \\
\cline{2-2}
 & \qquad\qquad 2x - 4 \\
 & \qquad\qquad -2x + 4 \\
\cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrr}
 & 1 & -4 & 6 & -4 \\
2 & & 2 & -4 & 4 \\
\hline
 & 1 & -2 & 2 & 0
\end{array}
$$

Either way, we see that $x^3 - 4x^2 + 6x - 4 = (x-2)(x^2 - 2x + 2)$.

All we have to do, in order to complete the problem, is to solve $x^2 - 2x + 2 = 0$, which can be easily done using the quadratic formula (note that it is impossible to factor this using rational numbers!).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(1)(2)}}{(2)(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{4 \pm \sqrt{-4}}{2}, \text{ Which is not a real number.}$$

So the only real root of $x^3 - 4x^2 + 6x - 4 = 0$ is $x = 2$.

Exercises

Solve the following polynomial equations.
You may have to,

1. Find the "candidate" rational solutions (using the rational root test)
2. Check which numbers you found above are actually roots of the polynomial (using factor theorem)
3. Divide out the factors (using long division or synthetic division)
4. Find the other real solutions using the quadratic formula

(a) $x^3 - 3x^2 + 4x - 2 = 0$ (b) $2x^3 - 2x^2 - 3x + 1 = 0$ (c) $4x^3 + 12x^2 + 11x + 3 = 0$

(d) $x^3 + x^2 - x - 1 = 0$ (e) $x^3 + 3x^2 + 3x + 1 = 0$ (f) $4x^4 + 8x^3 + 3x^2 - 2x - 1 = 0$

You could also follow the steps below,

1. Find the "candidate" rational solutions (using the rational root test)
2. Find one root out of the candidate solutions found above (using factor theorem)
3. Divide out the factor (using long division or synthetic division)
4. Using the rational root test, find the candidate rational solutions for the new polynomial you found in the previous steps after dividing out the factor.
5. Find one root out of the candidate solutions found for the reduced polynomial
6. Repeat 3 and 4 until you have found all the rational solutions
7. If the number of rational solutions is less than the degree of the original polynomial, it means that there will be some more roots which could be irrational or may be even not real.
8. Find the irrational solutions (if exists) using the quadratic formula

Note: Depending on the problem, one approach could be easier than the other.