MATH1550: Precalculus

Lecture 03

Monday, August 30, 2010

Recap of last class

We talked about

1. Evaluating mathematical expressions
2. Exponents: positive and negative integer exponents, zero exponent
3. n^{th} Roots; Principal n^{th} Roots

Review of n^{th} Roots

Given a real number a, the "n^{th} root of a" is a number b such that

$$b^n = a$$

Then we call b to be an n^{th} root of a.

- Both $1/4$ and $-1/4$ are *square* roots (i.e. the second roots) of $1/16$, because, $4^2 = 16$ and $(-4)^2 = 16$.
- 3 is a *cube* roots (i.e. the third root) of 27, because, $3^3 = 27$; in fact, 3 is the only real cube root of 27.
- -0.2 is a 5^{th} root -0.00032, because, $(-0.2)^5 = -0.00032$; in fact, -0.2 is the only real 5^{th} root of -0.00032.
- Both 2 and -2 are 4^{th} roots (i.e. the second roots) of 16, because, $2^4 = 16$ and $(-2)^4 = 16$. In fact, these are the only two real 4^{th} roots of 16
- -4 does not have a real square root because there is no real number a such that $a^2 = -4$

Principal n^{th} Root

We can make a few observations from the previous examples:

- Negative real numbers does not have real even roots
- Even numbered roots of positive real numbers always occur in pairs; one positive and one negative
- (Real valued) odd numbered roots are unique, and odd numbered root of a positive number is positive, and and odd numbered root of a negative number is negative.

To avoid the ambiguity of the even numbered roots of positive real numbers, we introduce the notion of **principal** n^{th} root. The principal n^{th} root, for even n, is simply the positive n^{th} root. For other cases, there is no ambiguity. So, the principal root and the real root are the same.

The principal n^{th} root is denoted by the "radical" sign $\sqrt[n]{}$
The principal square root is denoted just by $\sqrt{}$

Principal n^{th} Root: Examples

- $\sqrt{25} = 5$ because $5^2 = 5 \times 5 = 25$, and 5 is the principal square root of 25.
- Eventhough $(-5)^2 = 25$, it is WRONG to write $\sqrt{25} = -5$ according to the definition of $\sqrt{}$; since -5 is not the principal square root of 25.
- $\sqrt[4]{16} = 2$ because $2^4 = 16$, and since 2 is the principal 4^{th} root of 16.
- Eventhough $(-2)^4 = 16$, it is WRONG to write $\sqrt[4]{16} = -2$ according to the definition of $\sqrt[4]{}$; since -2 is not the principal 4^{th} root of 16.
- $\sqrt[3]{-125} = -5$ because $(-5)^4 = 125$, and it is unique.

If a has an n^{th} root, the principal n^{th} root of a is the root having the same sign as a.

CAUTION!!!

We can see that $\left(\sqrt[n]{a}\right)^n = a$ for any real number (if $\sqrt[n]{a}$ exists).

On the other hand, note that $\sqrt{(-6)^2} = \sqrt{36} = 6$ and in NOT -6.
But $\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$.

That means, $\sqrt[n]{(\quad)}$ and $(\quad)^n$, in general, are NOT the opposite operations like $\times$ and $\div$ or $+$ and $-$.

In fact, for any real number (positive or negative) a, and even n, $\sqrt[n]{a^n}$ will be equal to a, but without the sign, if a is negative.
In notation we write

$$\sqrt[n]{a^n} = |a|.$$

$|a|$ is called the **absolute value** or the **modulus** of a.

A bit on $|a|$

$|a|$ is identical to a, if a is positive;

and if a is negative, $|a|$ will have the same value as a, but without the sign.

For example $|-5.236| = 5.236$; and $|6.8153| = 6.8153$. As a convention we write $|0| = 0$.

The absolute value is the “distance” (in numbers) from 0 to a given number.

Laws of principal roots

Remember the following properties of the principal roots ...

Let a and b be two real numbers, and n and m be two integers,

1. $\sqrt[m]{ab} = \sqrt[m]{a}\,\sqrt[m]{b}$
2. $\sqrt[m]{\dfrac{a}{b}} = \dfrac{\sqrt[m]{a}}{\sqrt[m]{b}}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Some thing more than just a notation

We denote the *principal* n^{th} root of a real number a by $a^{1/n}$. Therefore,

$$\sqrt[n]{a} = a^{1/n}$$

This is more than just a notation. We can actually *treat* the principal n^{th} root as a "fractional power".

CAUTION:
Keep in mind that $a^{1/n}$ is the principal n^{th} root of a real number!

Review on Rational Exponents

With the notation introduced above, we immediately realize the notion of rational exponents:
For a real number a and two integers m and n (assume bot are nonzero), consider $\left(\sqrt[n]{a}\right)^m$; assuming that $\sqrt[n]{a}$ makes sense.

With the notation of principal n^{th} root, we may write the "root" as an exponent: $\left(\sqrt[n]{a}\right)^m = \left(a^{1/n}\right)^m$.

Then treating the exponent $1/n$ as a regular exponent, and by the laws of exponents: $\left(\sqrt[n]{a}\right)^m = \left(a^{1/n}\right)^m = a^{m/n}$.

Then we may reduce m/n to the smallest terms possible.

This is the whole idea of **Rational Exponents**; it means that the exponent is a rational number!

Just like we consider integers as rational numbers, we may consider integer exponents and principal roots as rational exponents.

Try it yourself...

Simplify the following rational exponents

1. $32^{2/5}$
2. $0.01^{-3/2}$;
   (Hint: First write 0.01 as an exponent of an easy number)
3. $\sqrt[3p]{8^{2p}}, p \neq 0$;
   (Hint: (1) Rational exponent and (2) laws of exponents)
4. $(256^{-3/4})^{8/6}$;
   (Hint: (1) $256 = 2^8$ and (2) laws of exponents)

Laws of Exponents - Revisit

Remember the following rules...

Let a and b be two real numbers, and n and m be two **rational numbers**,

1. $a^m a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $\dfrac{a^m}{a^n} = a^{m-n}$
4. $(ab)^m = a^m b^m$

Answers

Simplify the following rational exponents

1. $32^{2/5} = (2^5)^{2/5} = 2^{5(2/5)} = 2^2 = 4$
2. $0.01^{-3/2} = (10^{-2})^{-3/2} = 10^{(-2)(-3/2)} = 10^3 = 1000$
3. $\sqrt[3p]{8^{2p}} = (8^{2p})^{1/(3p)} = 8^{(2p)/(3p)} = 8^{2/3} = (2^3)^{2/3} = 2^{3(2/3)} = 2^2 = 4$
4. $(256^{-3/4})^{8/6} = 256^{(-3/4)(8/6)} = 256^{-1}$

A story I forgot to tell you

Why is the zeroth power of any non zero real number is 1?

Pf.
Let a be any non zero real number. Then, obviously, for any integer n,

$$\frac{a^n}{a^n} = 1. \tag{1}$$

If we write the division using the rules of exponents, we may write:

$$\frac{a^n}{a^n} = a^{n-n} = a^0. \tag{2}$$

Since (1) and (2) are just two ways of writing the same thing, we MUST have

$$a^0 = 1.$$

Q.E.D.

This demonstrates how you would write a simple proof. Abbreviations "pf." stands for "proof" and "Q.E.D. stands for the Latin phrase "quod erat demonstrandum", which means "that which was to be demonstrated".