MATH1550: Precalculus

Lecture 04

Tuesday, August 31, 2010

Recap of last class

Over the last two classes, we primarily reviewed the concept of exponents, roots and rational exponents.

Plan for the next few days

I hope to quickly go through chapter 1 of the textbook. I am afraid, we cannot spend more than 3 or 4 days on chapter 1. But we will cover the important points.

Apologies in advance :(

Sections to be Read from the Textbook

1. §1.1: Set of real numbers
2. §1.2: Absolute value
3. §1.3: Solving equations (PLEASE read the note on "Order of Precedence and Factoring" before this section)
4. §1.4: Rectangular coordinates & Visualizing data.
5. §1.5: Graphs and Graphing Utilities
6. §1.6: Equations of lines
7. §1.7: Symmetry and Graphs & Circles

The Set of Real Numbers

Hope you remember the discussion on Real numbers. It was the first thing we started with. There we talked about:

1. Natural (Counting) numbers ($\mathbb{N}$):
   (0), 1, 2, 3, ... ,687998, ...
2. Integers ($\mathbb{Z}$):
   ..., -98709079, ... -2,-1,0,1,2,...,897098,...
   These include the Natural numbers
3. Rational numbers ($\mathbb{Q}$):
   Ratios of Integers of the form p/q, where p and q are integers, and q is nonzero
   These include the Integers (remember why?)
4. Irrational numbers:
   Numbers which cannot be expressed as the ratio of two integers
5. Real numbers:
   Collection of both rational and irrational numbers.

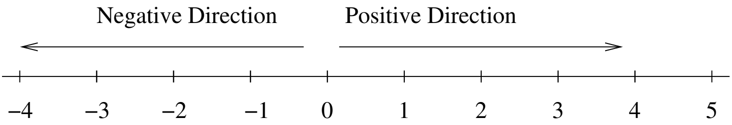
Graphical Representation of Numbers: The Real Line

We can represent real numbers graphically as a distance along a strait line, from a fixed point called the **origin**, which represents "zero" (0). The origin represents 0 for an obvious reason: it has zero distance from the origin!

Since we can go in two directions from the origin, call on side "positive" and the other "negative".

This chose is purely arbitrary. As a convention, we take the right hand side from the origin or the segment going up from the origin as positive, and the other side negative.

The **absolute value** of a number is the distance to that number from the origin, disregard of the direction.



$<$, $>$, $\leq$ and $\geq$

This is how we read them out ...

$a < b$ "a is less than b" (automatically means b is greater than a)
a is to the **left of** (or **below**) b, on the number line.

$a \leq b$ "a is less than or equal to b" (automatically means b is greater than or equal to a)

$a > b$ "a is greater than b" (automatically means b is less than a)
a is to the **right of** (or **above**) b, on the number line.

$a \geq b$ "a is greater than or equal to b" (automatically means b is less than or equal to a)

These four symbols ($<$, $>$, $\leq$ and $\geq$) are called binary relations on the real numbers. Equal sign $=$ also denotes a binary relation.
These are called binary relation because they can be used to compare the location of two numbers a and b, with respect to each other.

$<$, $>$, $\leq$ and $\geq$ are used to define an order on the real numbers (helps us to say what's bigger and what's smaller)

More on $<$, $>$, $\leq$ and $\geq$

We can write the possible range of a variable x using $<$, $>$, $\leq$ and $\geq$:

$a < x$	a is less than x; equivalently, x is greater than a; meaning x can take any value greater than a
$x > a$	x is greater than a. Has the same meaning as $a < x$.
$a \leq x$	a is less than or equal to x; equivalently, x is greater than or equal to a; meaning x can take any value greater than or equal to a
$x \geq a$	x is greater than or equal to a. Has the same meaning as $a \leq x$.

Now you should be able to guess what does $x < a$, $x \leq a$, $a > x$ and $a \geq x$ means.

Further more More on $<$, $>$, $\leq$ and $\geq$

We can combine up to two statements containing $<$, $>$, $\leq$ and $\geq$:

$a < x < b$ means $a < x$ **and** $x < b$

$a < x \leq b$ means $a < x$ **and** $x \leq b$

$a \leq x < b$ means $a \leq x$ **and** $x < b$

$a \leq x \leq b$ means $a \leq x$ **and** $x \leq b$

We usually do not write $a \leq x \geq b$, $a \geq x \leq b$, $a < x \geq b$, $a \leq x > b$, $a > x \leq b$ or $a \geq x < b$.

Some thing to think about ...

For some constant a, and a variable x, if we are told that $a \leq x \leq a$, what can we conclude about x?

On the other hand, if we are told that $a < x < a$, what can we conclude about x?

How about $a < x \leq a$ or $a \leq x < a$?

Some Obvious Properties of $<$, $>$, $\leq$ and $\geq$

If we are given three real numbers a, b an c, and if we have $a < b$ and $b < c$, then we see that $a < b$.
In words, " If a is smaller than b, and if b is smaller than c, then a is smaller than c".

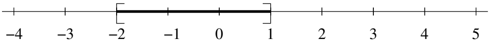
Can you how it works for $>$, $\leq$ and $\geq$?

Interval Notation

The so called interval notation is a compact way to denote a region on the number line.

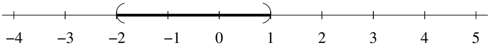
$[a, b]$ means all the numbers on the real line between a and b, including a and b. i.e. all x, such that $a \leq x \leq b$. Often called "the **closed** interval a to b".

Example: $[-2, 1]$ or equivalently, $-2 \leq x \leq 1$



(a, b) means all the numbers on the real line between a and b, NOT including a and b. i.e. all x, such that $a < x < b$. Often called "the **open** interval a to b".

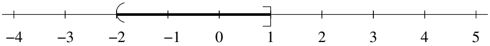
Example: $(-2, 1)$ or equivalently, $-2 < x < 1$



Mixed Intervals

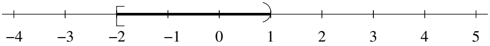
$(a, b]$ means all the numbers on the real line between a and b, NOT including a but including b. i.e. all x, such that $a < x \leq b$.

Example: $(-2, 1]$ or equivalently, $-2 < x \leq 1$



$[a, b)$ means all the numbers on the real line between a and b, including a but NOT including b. i.e. all x, such that $a \leq x < b$.

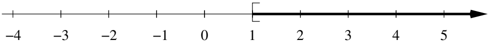
Example: $[-2, 1)$ or equivalently, $-2 \leq x < 1$



Unbounded Intervals to the Right

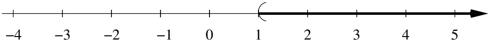
$[a, \infty)$ All the numbers on the real line greater than or equal to a,
i.e. all x, such that $a \leq x$.

Example: $[1, \infty)$ or equivalently, $1 \leq x$



(a, ∞) All the numbers on the real line greater than a,
i.e. all x, such that $a < x$.

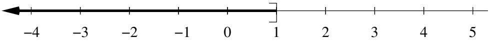
Example: $(1, \infty)$ or equivalently, $1 < x$



Unbounded Intervals to the Left

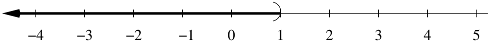
$(-\infty, a]$ All the numbers on the real line less than or equal to a, i.e. all x, such that $x \leq a$.

Example: $(-\infty, a]$ or equivalently, $x \leq 1$



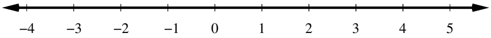
$(-\infty, a)$ All the numbers on the real line greater than a, i.e. all x, such that $x < a$.

Example: $(-\infty, 1)$ or equivalently, $x < 1$



Bidirectional Unbounded Interval

$(-\infty, \infty)$ All the numbers on the real line.



Absolute Value

The absolute value of a real number x is denoted by $|x|$.

Geometric interpretation

The absolute value of a real number x, denoted by $|x|$, is the distance from the origin to x (or x to the origin), regardless of the direction.

Algebraic interpretation

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

Practically, it all amounts to dropping the $(-)$ of a negative real number.

Properties of Absolute Value

1. For any real number x, we have
   1. $|x| \geq 0$
   2. $x \leq |x|$ and $-x \leq |x|$
   3. $|x^2| = x^2$
2. For any two real numbers a and b, we have
   1. $|ab| = |a||b|$ and if $b \neq 0$, $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
   2. $|a+b| \leq |a| + |b|$ (This is called the triangle inequality)

Can you explain why these are true?

closed You can use either the geometric interpretation or the algebraic interpretation to explain these.

Can you tell under what conditions we get $|x| = 0$, $x = |x|$, $-x = |x|$ and $|a+b| = |a| + |b|$?

Properties of Absolute Value

1. For any real number x, we have
   1. $|x| \geq 0$:
      This should obvious from the geometric interpretation. From the algebraic interpretation: if x is positive (i.e. $x > 0$) to start with, then $|x| = x$. Hence $|x| > 0$ If x is negative (i.e. $x < 0$), $|x| = -x$. Since negative of a negative is positive, $|x| > 0$. If $x = 0$, $|x| = 0$. Therefore, For any real number x, $|x| \geq 0$.
   2. $x \leq |x|$ and $-x \leq |x|$
      This can be explained using the algebraic interpretation: If $x < 0$ but $|x| = -x$. Therefore, for $x < 0$, $x < |x|$. If $x > 0$, $|x| = x > -x$. If $x = 0$, $|x| = 0 = x = -x$.
   3. $|x^2| = x^2$
      This too can be explained using the algebraic interpretation: For any nonzero real number x, $x^2 > 0$. Therefore, $|x^2| = x^2$. If $x = 0$ and $|x| = 0$. Hence, $x^2 = 0 = |x^2|$
2. For any two real numbers a and b, we have
   1. $|ab| = |a||b|$ and if $b \neq 0$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
   2. $|a + b| \leq |a| + |b|$ (This is called the triangle inequality)

Rewrite Expressions to Eliminate Absolute Value

KEY POINT: Break in to intervals!

Examples:

1. $|\pi - 4| + 1$
   Recall that $\pi = 3.14159265\ldots$. Then, $\pi - 4$ is negative (i.e. $\pi - 4 < 0$). Therefore, $|\pi - 4| = -(\pi - 4) = 4 - \pi$.
   Therefore, $|\pi - 4| + 1 = 4 - \pi + 1 = 5 - \pi$.
2. $|x - 5|$, given $x \geq 5$
   Since $x \geq 5$, $x - 5$ will never be negative. Therefore, $|x - 5| = x - 5$, when $x \geq 5$.
3. $|y + 3| + 3$, given $y < -3$
   Since $y < -3$, $y + 3$ is always negative. Therefore, $|y + 3| = -(y + 3) = -y - 3$. Hence,
   $|y + 3| + 3 = -y - 3 + 3 = y$, for $y < -3$.

Another Example: a MUCH harder one

Q: Rewrite $f(x) = |x+1| + |x-2| - 3$ without using the absolute values.

A: KEY POINT: Break in to intervals!

Step 1: Identify the necessary regions.

Note that for $x < -1$, $x+1$ will always be negative, and for $x < 2$, $x - 2$ will always be negative. We can construct the following table:

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x < 2$
$x+1$	Negative	0	Positive	Positive	Positive
$\lvert x+1 \rvert$	$-(x+1)$	$x+1$	$x+1$	$x+1$	$x+1$
$x-2$	Negative	Negative	Negative	0	Positive
$\lvert x-2 \rvert$	$-(x-2)$	$-(x-2)$	$-(x-2)$	$x-2$	$x-2$

Therefore, for $x < -1$:
$f(x) = -(x+1) + (-(x-2)) - 3 = -x - 1 - x + 2 - 3 = -2x - 2;$

for $-1 \leq x < 2$:
$f(x) = (x+1) + (-(x-2)) + 3 = x + 1 - x + 2 - 3 = 0;$

for $2 \leq x$:
$f(x) = (x+1) + (x-2) + 3 = x + 1 + x - 2 - 3 = 2x - 4.$

One more Example

Q: Rewrite $f(x) = |x+1| \cdot |x-2|$ without using the absolute values.

A: Quite similar to the previous example. The key is to identify the proper regions ans apply the algebraic interpretation of the absolute value.

Note that for $x < -1$, $x+1$ will always be negative, and for $x < 2$, $x-2$ will always be negative. We can construct the following table:

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x < 2$
$x+1$	Negative	0	Positive	Positive	Positive
$\lvert x+1 \rvert$	$-(x+1)$	$x+1$	$x+1$	$x+1$	$x+1$
$x-2$	Negative	Negative	Negative	0	Positive
$\lvert x-2 \rvert$	$-(x-2)$	$-(x-2)$	$-(x-2)$	$x-2$	$x-2$

Therefore, for $x < -1$:
$f(x) = (-(x+1))(-(x-2)) = (x+1)(x-2)$;

for $-1 \leq x < 2$:
$f(x) = (x+1)(-(x-2)) = -(x+1)(x-2)$;

for $2 \leq x$:
$f(x) = (x+1)(x-2) = (x+1)(x-2)$.

Distance on a number line

The distance between two numbers a and b on the real line is given by $|a - b|$ and this is equal to $|b - a|$.

Examples:

Distance between 5 and 8 is $|8 - 5| = |3| = 3$, or $|5 - 8| = |-3| = 3$.

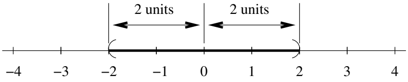
Distance between 5 and -2 is $|5 - (-2)| = |5 + 2| = |7| = 7$, or $|-2 - 5| = |-7| = 7$.

Displaying intervals defined by absolute values

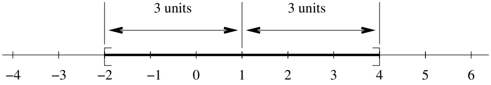
Think about the geometric interpretation of absolute values.

Examples:

$|x| < 2$ This represents the numbers which are less than 2 units of distance away from the origin. This means the region between -2 and 2. Therefore, the required region is $-2 < x < 2$.



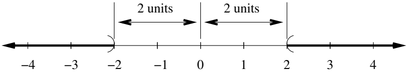
$|x - 1| \leq 3$ This represents the numbers which are less than 3 units of distance away from the 1 (recall the distance formula). This means the region between -2 (found by $1 - 3$) and 4 (found by $1 + 3$). Therefore, the required region is $-2 \leq x \leq 4$.



Displaying intervals defined by absolute values

More Examples:

$|x| > 2$ This represents the numbers which are more than 2 units of distance away from the origin. This means the region outside -2 and 2. Therefore, the required region is $2 < x$ or $x < -2$.



$|x - 1| \geq 3$ This represents the numbers which are greater than 3 units of distance away from the 1 (recall the distance formula). This means the region outside -2 (found by $1 - 3$) and 4 (found by $1 + 3$). Therefore, the required region is $-2 \geq x$ or $x \geq 4$.