MATH1550: Precalculus

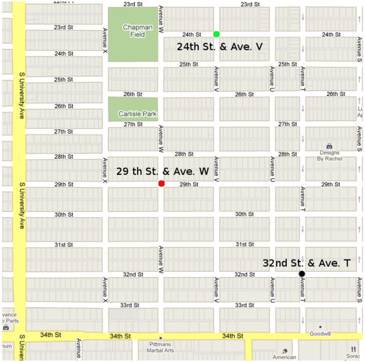
Lecture 06

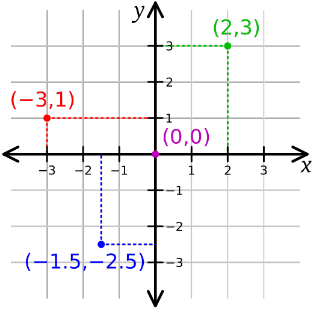
Tuesday, September 02, 2010

Recap of last class

Plan for today

Rectangular Coordinates



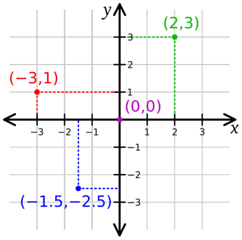


Rectangular (or Cartesian) **coordinates** are used to specify a location of a point in space with respect to a horizontal line (called the **x−axis**) and a vertical line (called the **y−axis**); just like you would talk about a location in Lubbock as the intersection of a street and an avenue.

The point of intersection of the $x-$ and $y-$ axes is called the **origin**.

The origin is usually, for obvious reasons, denoted by letter *0*.

Rectangular Coordinates



The location of points are denoted by listing first the *distance along the* $x-$*axis* (to the right) followed by the *distance along the* $y-$*axis* (up). For example the point shown in green has **coordinates** (2,3). Conversely, (2,3) means that, if you go 2 units to the right (along the $x-$axis) and 3 units up (along the $y-$axis), you will end up on the green point.

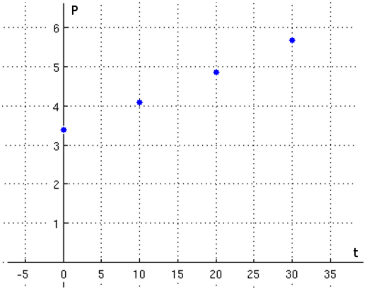
Since we take the distance to the right along the $x - axis$ as positive, the distance to the left along the $x-$ axis is taken as negative. Similarly, since we take the distance up along the $y - axis$ as positive, the distance down along the $y-$ axis is taken as negative.
As such, the red colored point has coordinates (-3,1) because it is 3 units to the left and 1 unit up.
The blue colored point has coordinates (-1.5,-2.5) because it is 1.5 units to the left and 2.5 unit down.
The origin shown in purple has coordinates (0,0). (WHY?)

Representing data on a rectangular coordinate system

Year	Population (Billions)
1965	3.345
1975	4.086
1985	4.850
1995	5.687

Data from the textbook, p.23; U.S. Census Bureau

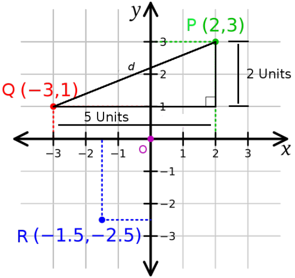
t	P
0	3.345
10	4.086
20	4.850
30	5.687



Since we are interested in studying the variation of population over the years, we represent the "Year" by the $x-$axis, and the population by the $y-$axis.
If we try to represent the "zeroth year" abd "zero population" by the origin, we will have an awkward looking graph. Therefore, we can plot the "time since 1965" and population, to have a better looking graph.
Now we can plot the variation of population (abbreviated 'P') with the "years since 1965" (abbreviated 't') on the coordinate axis.
Each point shown on the graph has 't' units left and 'P' units up.

Distance between two points on a rectangular coordinate system

Given any two points on a rectangular coordinate system, note that we can construct a **Right angled Triangle**. As soon as we haw a right angled triangle, we can use the famous **Pythagorean Theorem** to find the distance between two points.



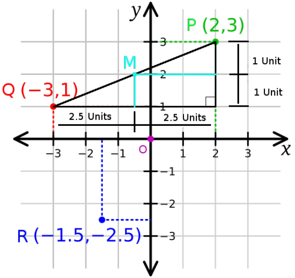
The distance d between points P and Q is:

$$
\begin{aligned}
d &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \\
d &= \sqrt{(-3-2)^2 + (1-3)^2} \\
&= \sqrt{5^2 + 2^2} \\
&= \sqrt{25+4} \\
\therefore\ d &= \sqrt{29}
\end{aligned}
$$

Usually, the line segment joining a point P and a point Q is written as line segment PQ.

Can you find the lengths of the line segments PR, OR OP, RQ, and RR?

Midpoint rule

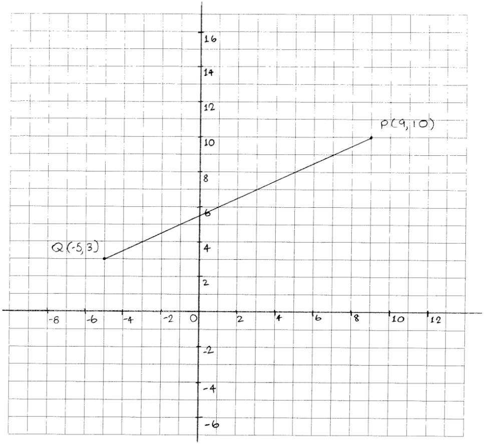


The midpoint M of the line segment PQ is given by

$$\begin{aligned} M &= \left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right) \\ &= \left(\frac{2 + (-3)}{2}, \frac{3 + 1}{2}\right) \\ &= \left(\frac{-1}{2}, \frac{4}{2}\right) \\ \therefore\ M &= (-0.5, 2) \end{aligned}$$

Can you find the midpoints of the line segments PR, OR OP, RQ, and RR?

How can you divide a line segment to a given ratio?



Given the points P and Q here, can you find the coordinates of the point R, such that $\dfrac{\text{Length of RQ}}{\text{Length of PR}} = \dfrac{5}{2}$?

How can you divide a line segment to a given ratio? Method

Think about a much simpler problem: If you are given two numbers a and b, (assume $b > a$) how would you find number c such that $\frac{c-a}{b-c} = \frac{p}{q}$?

You can solve this simpler problem in two steps...

1. First convert the ratio to a fraction of the total length $b - a$:
   Note that $c - a = (b - a)\frac{p}{(p+q)}$
2. Then, take a over to the right hand side:
   We get $c = a + (b - a)\frac{p}{(p+q)}$

Numerical example:

Given the two numbers 3 and 21, find the number x such that $\frac{x-3}{21-x} = \frac{2}{7}$.

1. First convert the ratio to a fraction of the total length $21 - 3 = 18$:
   Note that $x - 3 = (21 - 3)\frac{2}{(2+7)} = \frac{(18)(2)}{9} = 4$
2. Then, take a over to the right hand side
   Hence, we get $x = 3 + 4 = 7$

CHECK THE SOLUTION: $x - 3 = 7 - 3 = 4$ and $21 - x = 21 - 7 = 14$. So, $\frac{x-3}{21-x} = \frac{4}{14} = \frac{2}{7}$

How can you divide a line segment to a given ratio? Method cont'd...

Extend the same argument to solve the original problem. This time, working on the $x-$ and $y-$coordinates separately.

Restate problem in numbers and words: Given $P = (9, 10)$ and $Q = (-5, 3)$, Find $R = (x_r, y_r)$ such that the ratio of length of line segments $\frac{RQ}{PR} = \frac{5}{2}$.

First find x_r:

Since $x_p = 9 > -5 = x_q$, we must have $\frac{(x_r - x_q)}{(x_p - x_r)} = \frac{5}{2}$. (If by some chance, we had $x_p < x_q$, we should write $\frac{(x_q - x_r)}{(x_r - x_p)}$)

Therefore, $x_p - x_r = 9 - x_r = (9 - (-5))\frac{2}{(2+5)} = \frac{(14)(2)}{7} = 4$, hence $9 - x_r = 4$. Solving for x_r, we get $x_r = 5$.

(Similarly, we could have worked with $x_r - x_q$ and get the same answer. I leave it for you to do this as an exercise.)

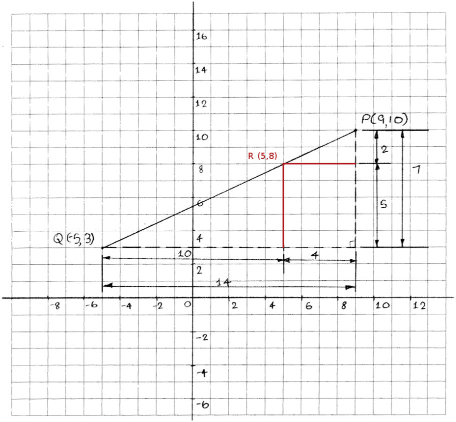
Then find y_r:

Since $y_p = 10 > 3 = y_q$, so we must write $\frac{(y_r - y_q)}{(y_p - y_r)} = \frac{5}{2}$.

Therefore, $y_p - y_r = 10 - x_r = (10 - 3)\frac{2}{(2+5)} = \frac{(7)(2)}{7} = 2$, hence $10 - y_r = 2$. Solving for y_r, we get $y_r = 8$.

Therefore, the coordinates of the required point $R = (5, 8)$.

How can you divide a line segment to a given ratio? SOLUTION



Method Summary: Divide the horizontal and vertical difference of coordinates to the given ratio.

How can you divide a line segment to a given ratio? Check...

$$\begin{aligned}
\text{Length of segment PR} &= \sqrt{(x_p - x_r)^2 + (y_p - y_r)^2} \\
&= \sqrt{(9-5)^2 + (10-8)^2} \\
&= \sqrt{4^2 + 2^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20}
\end{aligned}$$

$$\begin{aligned}
\text{Length of segment RQ} &= \sqrt{(x_r - x_q)^2 + (y_r - y_q)^2} \\
&= \sqrt{(5-(-5))^2 + (8-3)^2} \\
&= \sqrt{10^2 + 5^2} \\
&= \sqrt{100+25} \\
&= \sqrt{125}
\end{aligned}$$

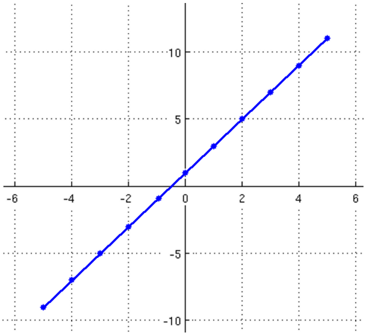
$$\frac{\text{Length of segment RQ}}{\text{Length of segment PR}} = \frac{\sqrt{125}}{\sqrt{20}} = \sqrt{\frac{125}{20}} = \sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$$

Graphing Mathematical Expressions

Given a mathematical expression $p(x)$ of a variable x, we can compute the value of $p(x)$ corresponding to a "few" values of x and use them to plot the graph of $p(x)$ versus x.

Example: Draw the graph of $p(x) = 2x + 1$

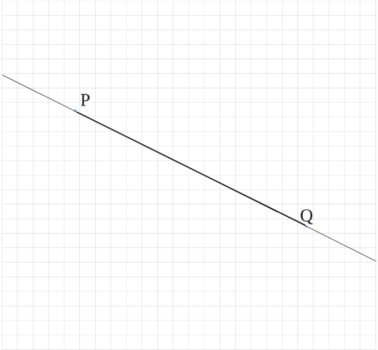
x	$p(x)$
-5	-9
-4	-7
-3	-5
-2	-3
-1	-1
0	1
1	3
2	5
3	7
4	9
5	11



Lines

Physical Interpretation

The **shortest path** between two distinct points is a **line**.
In other words, given two distinct points, we can draw a unique line joining them.



Some Terminology

Slope or Gradient (notation: m)

How much tilted or inclined a line is compared to the $x-$axis.
"How much does a line go **up** when it goes **one unit** to the **right**?"

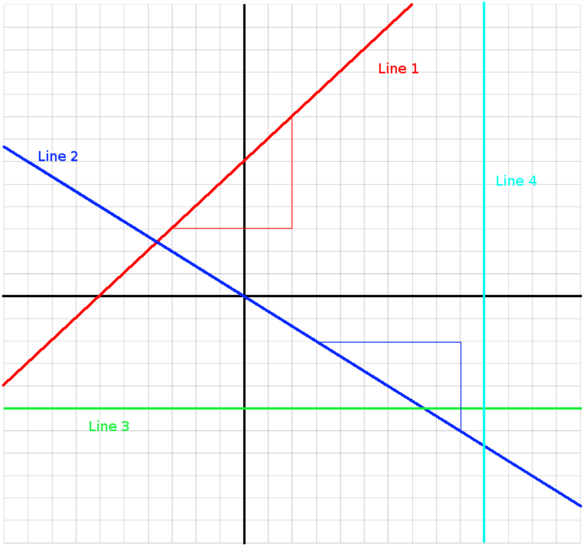
x Intercept

Where a line cuts (i.e. intersects) the $x-$axis.
"What is the $x-$value when $y = 0$?"

y Intercept (notation: c)

Where a line cuts (i.e. intersects) the $y-$axis.
"What is the $y-$value when $x = 0$?"

Can you find the Slope and $x-$ and $y-$ intercepts ?



Equations of lines

We can draw a unique strait line if we know:

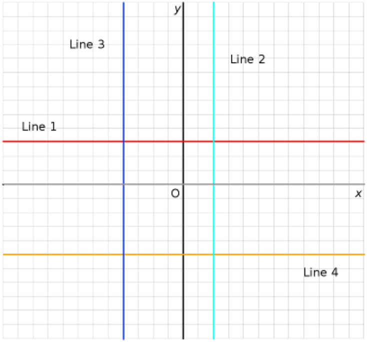
1. two distinct points the line goes through
2. a point the line goes through and its slope

It is very convenient to use an intercept (usually the $y-$intercept) as "the point" in the second option.

If we can draw a unique line when we are given one of these two cases, we can write down a mathematical equation for line in these two cases!

A nice observation about lines: "The slope of a given line never changes from point to point"

Equations of lines: Horizontal or Vertical



Note that the the $y-$value does not change with the $x-$value for horizontal lines and the $x-$value does not change with the $y-$value for vertical lines. Therefore, we can specify a horizontal line uniquely using the $y-$value and a vertical line uniquely by its $x-$value.

For example, the equation of line 1 will be $y = 3$. How about the others?

What will be the equations of the $x-$ and $y-$ axes?

Understanding the slope of a line

Slope tells "how many units a line goes **up**, when it goes **one unit to the right**"
In particular, if the slope is m, then, the line goes up m units, when it goes one unit to the right.

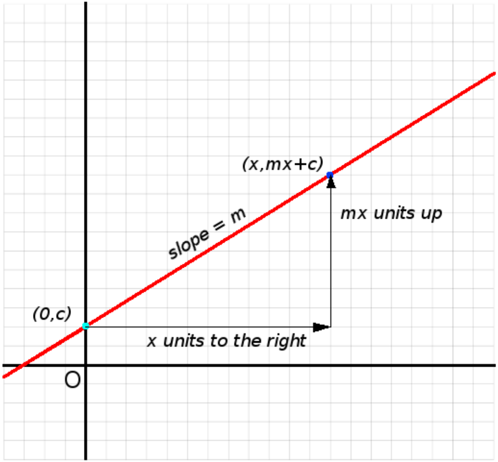
If a line has $m = 2$, then it goes 2 units up for each unit it goes to the right.
Equivalently, it goes 2 units down for each unit it goes to the left.

If a line has $m = -3$, then it goes -3 units up (i.e. 3 units down) for each unit it goes to the right.
Equivalently, it goes 3 units up for each unit it goes to the left.

If a line has $m = 0$, then it goes 0 units up for each unit it goes to the right (i.e. continues to run horizontal).

Finding the equation of a line

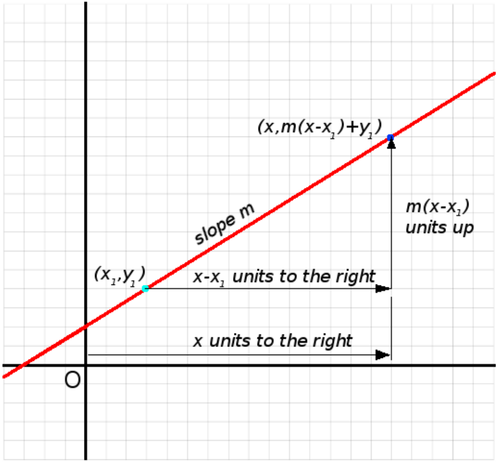
Given: The $y-$intercept (c) and the slope (m).



$$y = mx + c$$

Finding the equation of a line

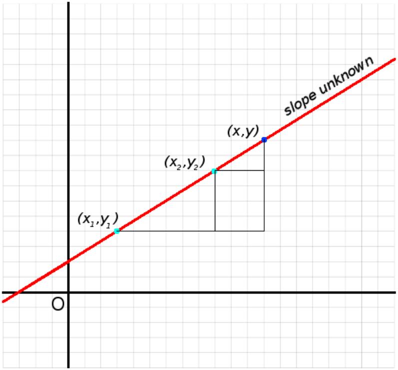
Given: A point (x_1, y_1) the line goes through and the slope (m)



$$y = m(x - x_1) + y_1$$

Finding the equation of a line

Given: Two points (x_1, y_1) and (x_2, y_2) the line passes through.



We do not know the slope but we can calculate it using the data:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Then using the previous method to write the equation: $y = m(x - x_1) + y_1$

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

General Equation of a Line

The "slope-intercept" version ($y = mx + c$) is the most famous form of representing the equation of a line. But it's not possible represent vertical lines using this form (WHY?).

$Ax + By + C = 0$ is called the general equation of a line, because, it can be used to represent ANY straight line.

When $B \neq 0$, we can write

$$y = \left(\frac{-A}{B}\right) x + \left(\frac{-C}{B}\right) \tag{1}$$

Which is in the slope-intercept form, with $m = -A/B$ and $c = -C/B$.

When $B = 0$, we can write

$$x = \frac{-C}{A} \tag{2}$$

Which is in the equation of a vertical line.

Parallel and Perpendicular Lines

- Two lines are parallel if they have the same slope
- Two lines are perpendicular if $m_1 = -1/m_2$, where m_1 and m_2 are the slopes of the two lines. (i.e. the slope of one line should be the negative reciprocal slope of the the other)

Examples:

- Two vertical lines are parallel and two horizontal lines are parallel
- A horizontal line and a vertical line are perpendicular
- The two lines $x + 2y + 1 = 0$ and $2x - y + 5 = 0$ are perpendicular
- The two lines $y = 3x + 4$ and $9x - 3y + 4 = 0$ are parallel

Exercise:
Given the line $2x - y + 1 = 0$

1. Find the line which is parallel to it and passes through the origin
2. Find the line which is perpendicular to it and passes through through the point $(1, -1)$
3. Can the line which passes through $(1, 1)$ and $(2, 2)$ be perpendicular or parallel to it?

Symmetry of Graphs

Symmetry about the $x-$axis

(x, y) and $(x, -y)$ are both points on the graph.
"Replacing y by $-y$ does not change the equation"

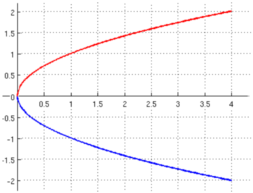
Symmetry about the $y-$axis

(x, y) and $(-x, y)$ are both points on the graph.
"Replacing x by $-x$ does not change the equation"

Symmetry about the origin

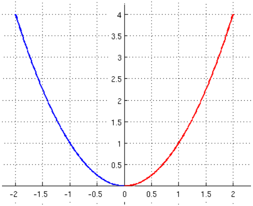
(x, y) and $(-x, -y)$ are both points on the graph.
"Replacing x by $-x$ **and** y by $-y$ together does not change the equation"

Symmetry of Graphs: Some Examples



$x = y^2$

Substituting $y = -y$;
$x = (-y)^2 = y^2$



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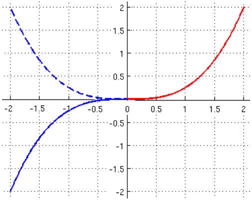

$y = x^3/4$

Substituting $x = -x$ and
$y = -y$;
$-y = (-x)^3/4$;
We get $-y = -x^3/4$;
Which becomes

$y = x^3/4$

Symmetry of Graphs: Another Example

Consider a graph defined by the equation $x^2 + y^2 = 4$. Then,

1. It is symmetric about the $x-$axis. Because, if we substitute $y = -y$, we get, $x^2 + (-y)^2 = 4$; which is identical to $x^2 + y^2 = 4$.
2. It is symmetric about the $y-$axis. Because, if we substitute $x = -x$, we get, $(-x)^2 + y^2 = 4$; which is identical to $x^2 + y^2 = 4$.
3. It is symmetric about the origin. Because, if we substitute $x = -x$, and $y = -y$ we get, $(-x)^2 + (-y)^2 = 4$; which is identical to $x^2 + y^2 = 4$.

Do you know what this equation describes?

Circle

A **circle** is the path of a point which moves at a constant distance (**radius**) away from a fixed point (**center**).
If (h, k) is the fixed point and if r is the radius, for any point (x, y) on the circle we should have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

If we square both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2$$

If we had $h = 0$, $k = 0$ (i.e. the center is the origin) and $r = 2$ we get the circle in the previous example.