MATH1550: Precalculus

Lecture 07

Quadratic Equations

Basic Problem: Given $ax^2 + bx + c = 0$, solve for x.

Factoring well help most of the time. Sometimes finding the factors may be hard.

It will be nice to have a formula :)

This is the famous "quadratic formula"

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

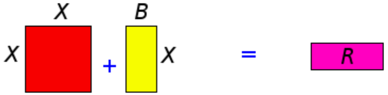
Motivation

All you know how to solve is $X^2 = r$, where r is any positive number.

The solution of this equation is $X = \pm\sqrt{r}$.

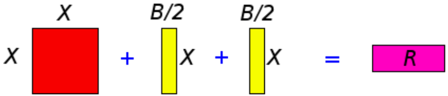
If we are given any equation, like $ax^2 + bx + c = 0$, we have to convert it to the above form $(X^2 = r)$. This technique is called "**completing the square**"

Completing the Square: In Pictures



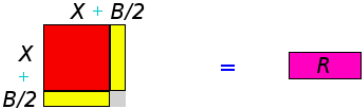
Given Equation

$X^2 + BX = R$

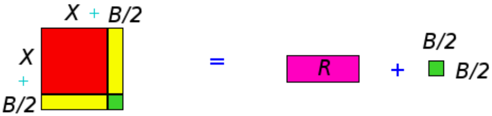


Split the BX term in to two parts:

$$\left(\frac{B}{2}\right)X + \left(\frac{B}{2}\right)X$$



We can arrange the left hand side to *almost* a square, except for a little piece missing.



Add the missing part $\left(\frac{B}{2}\right)^2$ to both sides of the equation. Now we have a **complete square** on the left side.

Completing the Square: In Numbers

$$\begin{aligned}
x^2 + 4x &= 12 \\
x^2 + 2x + 2x &= 12 \\
x^2 + 2x + 2x + \left(\frac{4}{2}\right)^2 &= 12 + \left(\frac{4}{2}\right)^2 \\
x^2 + 2x + 2x + 4 &= 12 + 4 \\
x^2 + 2x + 2x + 4 &= 16 \\
x(x+2) + 2(x+2) &= 16 \\
(x+2)(x+2) &= 16 \\
(x+2)^2 &= 4^2 \\
\text{Therefore, } x + 2 &= 4, \quad -4, \\
\text{Therefore, } x &= 2, \quad -6
\end{aligned}$$

Completing the Square: The General form $ax^2 + bx + c = 0$ with $a \neq 0$

$$
\begin{aligned}
ax^2 + bx &= -c \\
x^2 + \left(\frac{b}{a}\right)x &= -\frac{c}{a} \\
x^2 + \left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)x &= -\frac{c}{a} \\
x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
x\left(x + \frac{b}{2a}\right) + \left(\frac{b}{a}\right)\left(x + \frac{b}{2a}\right) &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) &= \frac{b^2 - 4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{(2a)^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Solve the Following

Solve the Following Equations using the Quadratic Formula, and find the sum and product of the roots.

1. $x^2 + 2x + 1 = 0$
2. $x^2 + 3x + 2 = 0$
3. $2x^2 - 5x = 0$
4. $4x^2 - 9 =$
5. $3x^2 - 4x + 1 = 0$
6. $x^2 + 3x - 4 = 0$

Some interesting Observations

The quadratic formula says that, given a quadratic equation $ax^2 + bx + c = 0$, ($a \neq 0$), the two solutions of x are $(-b + \sqrt{b^2 - 4ac})/(2a)$ and $(-b - \sqrt{b^2 - 4ac})/(2a)$:

Note the following:

1. $$\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{a}$$
2. $$\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}$$

The term $b^2 - 4ac$ under the square root sign can be used to determine the number of solutions of the equation. Hence it is called the "**discriminant**"

1. If $(b^2 - 4ac) < 0$, we **cannot** find any real solutions.
2. If $(b^2 - 4ac) = 0$, both solutions will be evaluated to -b/2a. (i.e. **coincident** solutions, **double** solutions,**multiplicity two**)
3. If $(b^2 - 4ac) > 0$, both solutions will have two **distinct** solutions

Using the discriminant

1. How many real solutions does the following equations have?
    1. $x^2 + x + 1 = 0$
    2. $x^2 - 8x + 16 = 0$
    3. $2x^2 + 3x - 1$
2. Find the all the values of k such that $x^2 + \sqrt{2}x + k = 0$ will have
    1. One real root
    2. Two real roots
    3. No real roots
3. Find the all the values of r such that $x^2 + rx + r = 0$ will have
    1. One real root
    2. Two real roots
    3. No real roots

Writing the Equations when Roots are Given

Examples:

1. Find the quadratic equations which has the roots $x = 2$ and 3.

   Ans. $(x - 2)(x - 3) = 0$

2. Find the quadratic equations which has the roots $x = 0$ and 4.

   Ans. $(x - 0)(x - 3) = 0$, which can be simplified to $x(x - 4) = 0$

3. Find the quadratic equations which has the roots $x = -1$ and 2.

   Ans. $(x - (-1))(x - 2) = 0$, which can be simplified to $(x + 1)(x - 2) = 0$

4. Find the quadratic equations which has the roots $x = a$ and b.

   Ans. $(x - a)(x - b) = 0$

Using the Quadratic Formula for Factoring

Example:

Factor $3x^2 + 7x + 2$ using the quadratic formula.

(Well, we know that we can factor it as $(3x+1)(x+2)$)

First set $3x^2 + 7x + 2 = 0$.

If we solve this using the quadratic formula we get the two solutions

$x = (-7 \pm \sqrt{7^2 - (4)(3)(2)})/((2)(3)) = (-7 \pm \sqrt{49-24})/6$
$= (-7 \pm \sqrt{25})/6 = (-7 \pm 5)/6.$

So we get the solutions $(-7-5)/6 = -12/6 = -2$ or $(-7+5)/6 = -2/6 = -1/3$

If we try to write down the equation which has -2 and $-1/3$ as solutions like before, we get

$(x-(-2))(x-(-1/3)) = (x+2)(x+(1/3)) = x^2 + (7/3)x + (2/3);$

so we are off by a factor of 3.

So, if we start by factoring the coefficient of the original x^2 term, we will have the exact factoring!

On the other hand, if you are given an equation with the coefficient of x^2 being 1, you can do the factorizing easily.

Using the Quadratic Formula for Factoring

Example:

1. $x^2 - 2x - 1$
   Set $x^2 - 2x - 1 = 0$, and solve for x. We get
   $x = (2 \pm \sqrt{4+4})/2 = (2 \pm 2\sqrt{2})/2 = 1 \pm \sqrt{2}$.
   So the factoring is $(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2}))$

2. $x^2 + 22x + 120$
   Set $x^2 - 2x - 1 = 0$, and solve for x. We get
   $x = (-22 \pm \sqrt{22^2 - (4)(120)})/2 = (-22 \pm \sqrt{484 - 480})/2$
   $= (-22 \pm \sqrt{4})/2 = (-22 \pm 2)/2 = (-11 \pm 1)/2$.
   So the factoring is $(x - (-12))(x - (-10))$; simplify to get
   $(x + 12)(x + 10)$

Find the center of a circle

Recall that the standard form of a circle of radius r and center (h,k) is $(x-h)^2+(y-k)^2=r^2$.
On the other hand, this can be simplified to get an equation of the form $ax^2+by^2+cx+dy+e=0$.
So, how can one go from one form to the other?
The answer is, completing the square:

$$\begin{aligned}
ax^2+ay^2+bx+cy+d &= 0\\
ax^2+bx+ay^2+cy+d &= 0\\
\left(x^2+\frac{b}{a}x\right)+\left(y^2+\frac{c}{a}y\right)+\left(\frac{d}{a}\right) &= 0\\
\left(x^2+\left(\frac{b}{a}\right)x+\left(\frac{b}{2a}\right)^2\right)+\left(y^2+\left(\frac{c}{a}\right)y+\left(\frac{c}{2a}\right)^2\right)+\left(\frac{d}{a}\right)-\left(\frac{b}{2a}\right)^2-\left(\frac{c}{2a}\right)^2 &= 0\\
\left(x+\left(\frac{b}{a}\right)\right)^2+\left(y+\left(\frac{c}{b}\right)\right)^2+\left(\frac{d}{a}\right)-\left(\frac{b}{2a}\right)^2-\left(\frac{c}{2a}\right)^2 &= 0
\end{aligned}$$

Therefore, $h=-\frac{b}{a}$, $k=-\frac{c}{a}$ and $r^2=\left(\frac{d}{a}\right)-\left(\frac{b}{2a}\right)^2-\left(\frac{c}{2a}\right)^2$
If the expression for r^2 is negative, then the given equation is not a valid circle.