MATH1550: Precalculus

Sections 2.3 & 2.4: Inequalities

Introduction

Given Problem:
Two mathematical expressions $p(x)$ and $q(x)$, related by $<$, $>$, $\leq$ or $\geq$.
e.g. $p_1(x) > q_1(x)$, $p_2(x) < q_2(x)$, $p_3(x) \leq q_3(x)$ or $p_4(x) \geq q_4(x)$.

Examples:

1. $x - 3 < 0$
2. $(x+1)(x+4) \geq x$
3. $\dfrac{3}{x} - 2 \leq 0$
4. $|2 - 3x| > 5$

Expected "Solution":
Find a region of values for the variable x so that the inequality is satisfied.

CAUTION!!!

- Consider the following examples:
  - Clearly, $-3 < -2$. If we multiply the numbers by -1, we get $3 > 2$. (i.e. Flipping the sign flips the inequality)
  - Clearly, $\frac{1}{4} < \frac{1}{2}$. If take the reciprocal, we get $4 > 2$. (i.e. Taking the reciprocal flips the inequality)
- The following operations will not change the inequality
  - Adding or subtracting the same quantity to both sides
  - Multiplying or dividing by a **positive, nonzero** quantity
  - Simplifying either side of of the inequality
- The following operations will flip the sign (i.e. $>$ with $<$, and $\geq$ with $\leq$)
  - Multiplying or dividing by a **negative, nonzero** quantity
  - Taking the reciprocal
- **Inequalities involving absolute values:** rewrite expressions to eliminate absolute value sign. This requires splitting in to intervals.

Solve the Following Inequalities

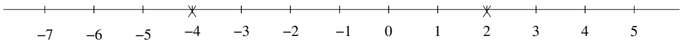
1. $2x - 7 < 11$ (p.110: ex. 2.3 - prob 1)
2. $6 - 4x \leq 22$ (p.110: ex. 2.3 - prob 2)
3. $\dfrac{3x}{5} - \dfrac{x-1}{3} < 1$ (p.110: ex. 2.3 - prob 5)
4. $0.99 < \dfrac{x}{2} - 1 < 0.999$ (p.110: ex. 2.3 - prob 11)
5. $|x - 4| \geq 4$ (p.110: ex. 2.3 - prob 20.b)
6. $\left|\dfrac{x-2}{3}\right| < 4$ (p.110: ex. 2.3 - prob 25)
7. $\left|\dfrac{3(x-2)}{4} + \dfrac{4(x-1)}{3}\right| \leq 2$ (p.110: ex. 2.3 - prob 28)
8. $|x - 1| + |x - 2| < 3$ (p.111: ex. 2.3 - prob 38)

Polynomial/Quotient Type Inequalities

Example: $(x-2)(x+4) > 0$

Step 1: Rewrite the inequality so that the polynomial or the quotient is on the left-hand side and the zero is on the right hand side.
Already done

Step 2: Find the key numbers (numbers at which the polynomial/quotient becomes zero or goes undefined), and locate them on the number line.
Set $(x-2)(x+4) = 0$ and solve for x: we get $x = 2$ or $x = -4$.



Step 3: List the intervals determined by the key numbers
$(-\infty, -4)$, $(-4, 2)$ and $(2, \infty)$

Step 4: Using a test point from each region, construct the **table of signs** and use it to identify the solution region(s)

Interval	Test Num.	$(x-2)$	$(x+4)$	$(x-2)(x+4)$
$(-\infty, -4)$	-5	*neg.*	*neg.*	*pos.*
-4	-4	*neg.*	0	0
$(-4, 2)$	0	*neg.*	*pos.*	*neg.*
2	2	0	*pos.*	0
$(2, \infty)$	3	*pos.*	*pos.*	*pos.*

Solution: $(-\infty, -4)$ **or** $(2, \infty)$

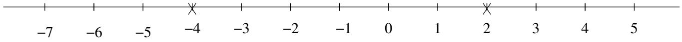
Polynomial/Quotient Type Inequalities

Example: $\dfrac{(x-2)}{(x+4)} \geq 0$

Step 1: Rewrite the inequality so that the polynomial or the quotient is on the left-hand side and the zero is on the right hand side. Done

Step 2: Find the key numbers (numbers at which the polynomial/quotient becomes zero or goes undefined), and locate them on the number line.

First note that the left hand side is not defined for $x = -4$. Set $\dfrac{(x-2)}{(x+4)} = 0$ and solve for x: we get $x = 2$. Key points are $x = -4$ and $x = 2$.



Step 3: List the intervals determined by the key numbers

$(-\infty, -4)$, $(-4, 2)$ and $(2, \infty)$

Step 4: Using a test point from each region, construct the **table of signs** and use it to identify the solution region(s)

Interval	Test Num.	$(x-2)$	$(x+4)$	$\dfrac{(x-2)}{(x+4)}$
$(-\infty, -4)$	-5	*neg.*	*neg.*	*pos.*
-4	-4	*neg.*	0	UNDEFINED
$(-4, 2)$	0	*neg.*	*pos.*	*neg.*
2	2	0	*pos.*	0
$(2, \infty)$	3	*pos.*	*pos.*	*pos.*

Solution: $(-\infty, -4)$ **or** $[2, \infty)$ (NOTE: 2 is included because of $\geq$)

Polynomial/Quotient Type Inequalities

Example: $(x+1)^2 \leq x$

Step 1: Rewrite the inequality so that the polynomial or the quotient is on the left-hand side and the zero is on the right hand side.
We have to simplify a bit:
Expand the square: $x^2 + 2x + 1 \leq x$; Subtracting x from both sides: $x^2 + x + 1 \leq 0$

Step 2: Find the key numbers (numbers at which the polynomial/quotient becomes zero or goes undefined), and locate them on the number line.
Look at the discriminant: $b^2 - 4ac = 1 - (4)(1)(1) = -3 < 0$. So, no solutions.
NO KEY POINTS = NO SIGN CHANGES

Step 3: List the intervals determined by the key numbers
$(-\infty, \infty)$ (WHY?)

Step 4: Using a test point from each region, construct the **table of signs** and use it to identify the solution region(s)

Interval	Test Num.	$x^2 + x + 1$
$(-\infty, \infty)$	0	*pos.*

Solution: NONE

No value of x will give the necessary inequality.

Solve the Following Inequalities

1. $x^2 + x - 6 < 0$ (p.120: ex. 2.4 - prob 9)
2. $(x-1)(x+3)(x+4) \geq 0$ (p.120: ex. 2.4 - prob 29)
3. $x(1-x^2)^4 + (x+3)(1-x^2)^4 \geq 0$ (p.120: ex. 2.4 - prob 42)
4. $\dfrac{2}{x} < \dfrac{x}{2}$ (p.120: ex. 2.4 - prob 54)
5. $\dfrac{x^2+3x}{x^2+8x+15} < 0$ (p.120: ex. 2.4 - prob 60)
6. $2x + 2 \geq (x+1)(x+4) \geq x$