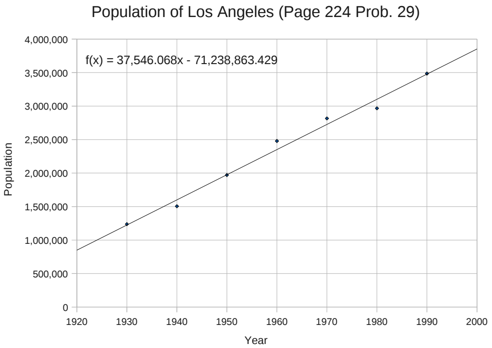
MATH1550: Precalculus

Chapter 4

Using Linear Functions (§4.1)

Year	Population
1930	1,238,048
1940	1,504,277
1950	1,970,358
1960	2,479,015
1970	2,816,061
1980	2,966,850
1990	3,485,398



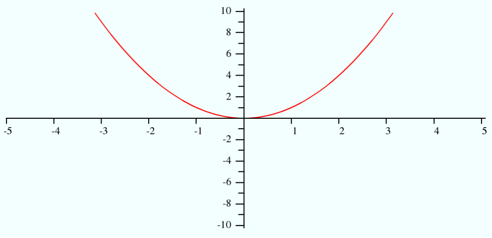
1. Use the regression line to compute an estimate for what the population of Los Angeles might have been in 2000. Then compute the percentage error in the estimate given the actual figure for 2000 is 3,694,820.
2. Find $f^{-1}(x)$
3. Use $f^{-1}(x)$ to estimate the year in which the population of Los Angeles might reach 4 million (4,000,000).

Quadratic Functions, $f(x) = ax^2 + bx + c, a \neq 0$ (§4.2)

Prerequisites:

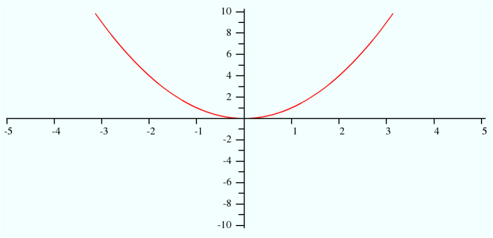
1. The graph of x^2
2. Completing the square
3. Translation and reflection of graphs

The graph of $y = x^2$ is called a **parabola**.

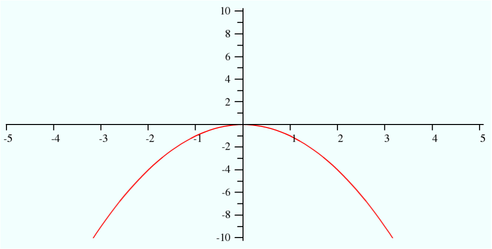


Graphs of $y = x^2$ and $y = -x^2$

The graph of $y = x^2$ opens up.



The graph of $y = -x^2$ opens down.



Review: Completing the square ...

Given $f(x) = ax^2 + bx + c$

1. Factor out a

$$f(x) = a\left(x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right)\right)$$

2. Add and subtract square of one half of the coefficient of the x term

$$f(x) = a\left(x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \left(\frac{c}{a}\right)\right)$$

3. Grouping

$$f(x) = a\left(\left[x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2\right] - \left(\frac{b}{2a}\right)^2 + \left(\frac{c}{a}\right)\right)$$

4. Simplify: The term in brackets $[\dots]$ is a perfect square

$$f(x) = a\left(\left[x + \left(\frac{b}{2a}\right)\right]^2 + \left(\frac{-b^2 + 4ac}{4a^2}\right)\right)$$

5. Multiply out by a and simplify more

$$f(x) = a\left(x + \left(\frac{b}{2a}\right)\right)^2 + \left(\frac{-b^2 + 4ac}{4a}\right)$$

Since we know the graph of $y = x^2$, we can sketch this now!

First Example

Given $f(x) = x^2 + 2x - 3$

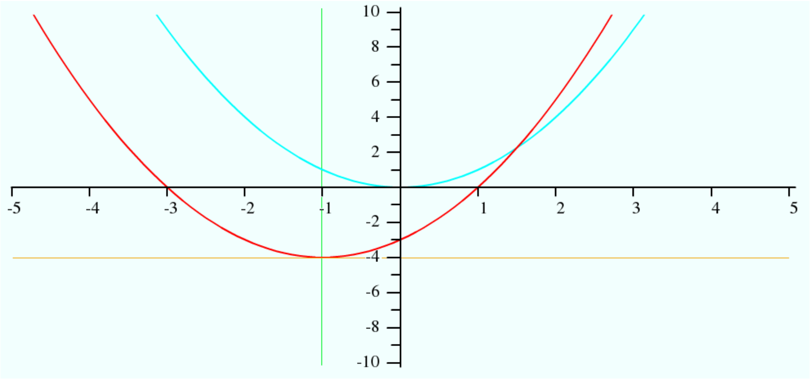
note that this can be factored: $f(x) = (x-3)(x+1)$

1. Factor out a
   $f(x) = x^2 + 2x - 3$
2. Add and subtract square of one half of the coefficient of the x term
   $f(x) = x^2 + 2x + 1 - 1 - 3$
3. Grouping
   $f(x) = \left(x^2 + 2x + 1\right) - 4$
4. Simplify: The term in brackets $[\ldots]$ is a perfect square
   $f(x) = (x+1)^2 - 4$
5. Multiply out by a and simplify more
   $f(x) = (x+1)^2 - 4$

Since we know the graph of $y = x^2$, we can sketch this now!

Shift 1 unit in the negative $x-$direction and shift 4 units down

First Example: The graph of $y = f(x) = x^2 + 2x - 3$



Observe the symmetry about the line $x = -1$

Some observations of the graph of $y = f(x) = x^2 + 2x - 3$

x-intercept: point where the graph crosses the x-axis.

Found by setting $y = 0$ and solving for x.

$x^2 + 2x - 3 = 0$; Factoring, we get $(x + 3)(x - 1) = 0$, so the solutions are $x = -3$ or $x = 1$; these are the $x-$intercepts.

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y-intercept: point where the graph crosses the y-axis.

Found by setting $x = 0$ and solving for y.

$y = (0)^2 + (2)(0) - 3 = -3$.

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maximum value: the range value that is greater than all other function values and the domain value that gives it.

Found by graphical observation.

Unbounded.

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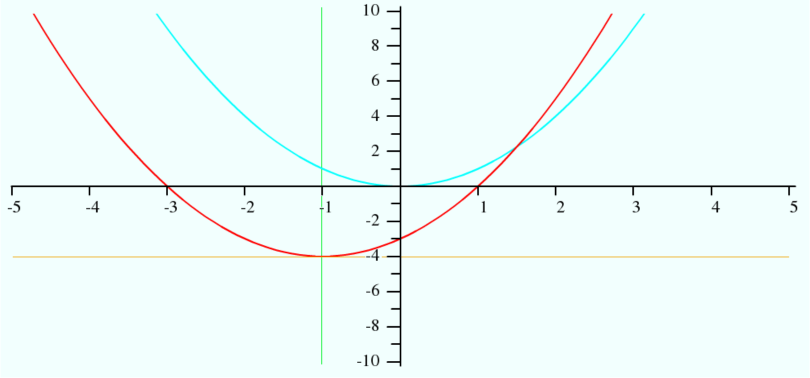
minimum value: the range value that is less than all other function values and the domain value that gives it.

Found by graphical observation.

Look at the result of completing the square: $f(x) = (x + 1)^2 - 4$. When we scan through the values of x, at $x = -1$, we get $x + 1 = -1 + 1 = 0$. Then $f(-1) = -4$. We cannot make $f(x)$ take any value less than that because any value of x, other than -1 will make $x + 1 > 0$.

First Example: The graph of $y = f(x) = x^2 + 2x - 3$

Factor form: $y = (x+3)(x-1)$; Complete square form: $y = (x+1)^2 - 4$



Symmetric about the line $x = -1$
$x-$intercepts : $x = -3$, $x = 1$
$y-$intercept : $y = -3$
Maximum: Unbounded
Minimum: -4

Second Example

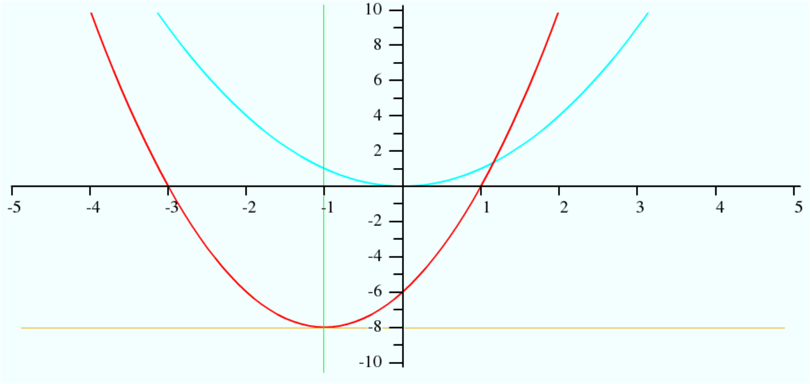
Given $f(x) = 2x^2 + 4x - 6$
note that this can be factored: $f(x) = 2(x-3)(x+1)$

1. Factor out a
   $f(x) = 2\left(x^2 + 2x - 3\right)$
2. Add and subtract square of one half of the coefficient of the x term
   $f(x) = 2\left(x^2 + 2x + 1 - 1 - 3\right)$
3. Grouping
   $f(x) = 2\left(\left[x^2 + 2x + 1\right] - 4\right)$
4. Simplify: The term in brackets $[\ldots]$ is a perfect square
   $f(x) = 2\left([x+1]^2 - 4\right)$
5. Multiply out by a and simplify more
   $f(x) = 2\,(x+1)^2 - 8$

Since we know the graph of $y = x^2$, we can sketch this now!

Second Example: The graph of $y = f(x) = 2x^2 + 4x - 6$

Factor form: $y = 2(x+3)(x-1)$; Complete square form: $y = 2(x+1)^2 - 8$



Symmetric about the line $x = -1$
$x-$intercepts : $x = -3$, $x = 1$
$y-$intercept : $y = -6$
Maximum: Unbounded
Minimum: -8

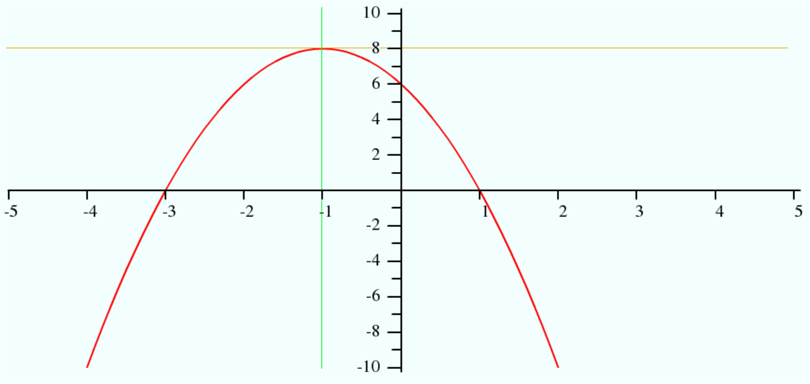
Third Example

Given $f(x) = -2x^2 - 4x + 6$
note that this can be factored: $f(x) = -2(x-3)(x+1)$

1. Factor out a
   $f(x) = -2\left(x^2 + 2x - 3\right)$
2. Add and subtract square of one half of the coefficient of the x term
   $f(x) = -2\left(x^2 + 2x + 1 - 1 - 3\right)$
3. Grouping
   $f(x) = -2\left(\left[x^2 + 2x + 1\right] - 4\right)$
4. Simplify: The term in brackets $[\ldots]$ is a perfect square
   $f(x) = -2\left([x+1]^2 - 4\right)$
5. Multiply out by a and simplify more
   $f(x) = -2(x+1)^2 + 8$

Third Example: The graph of $y = f(x) = -2x^2 - 4x + 6$

Factor form: $y = -2(x+3)(x-1)$; Complete square form: $y = -2(x+1)^2 + 8$



Symmetric about the line $x = -1$
$x-$intercepts : $x = -3$, $x = 1$
$y-$intercept : $y = 6$
Maximum: 8
Minimum: Unbounded

Fourth Example

Given $f(x) = x^2 + 2x + 2$

note that this cannot be factored: $b^2 - 4ac = 4 - (4)(1)(2) = 4 - 8 = -4 < 0$

1. Factor out a

   $f(x) = \left(x^2 + 2x + 2\right)$

2. Add and subtract square of one half of the coefficient of the x term

   $f(x) = \left(x^2 + 2x + 1 - 1 + 2\right)$

3. Grouping

   $f(x) = \left(\left[x^2 + 2x + 1\right] + 1\right)$

4. Simplify: The term in brackets $[\ldots]$ is a perfect square

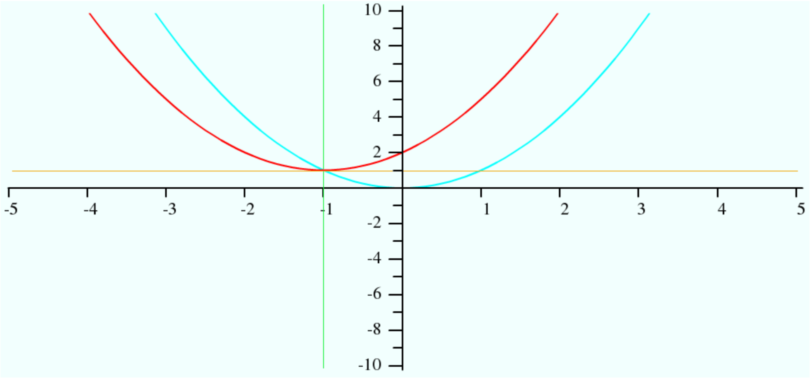
   $f(x) = \left([x+1]^2 + 1\right)$

5. Multiply out by a and simplify more

   $f(x) = (x+1)^2 + 1$

Fourth Example: The graph of $y = f(x) = x^2 + 2x + 2$

Complete square form: $y = (x+1)^2 + 1$



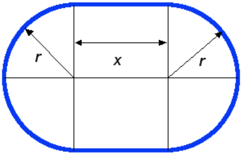
Symmetric about the line $x = -1$
$x-$intercepts : None
$y-$intercept : $y = 2$
Maximum: Unbounded
Minimum: 1

FOUR STEP process of problem solving.

1. **Understand:** Read the problem and restate it in your own words.
2. **Plan:** Sketch pictures and find equations.
3. **Carry Out:** Do the mathematics.
4. **Examine:** Consider your answer in relation to the problem. Does it make sense?

Examples

1. Find, if any, two consecutive even numbers which add up to 16.
2. The fixed cost of printing a book is \$1733.20 The printing cost per copy is \$12.61. If a copy is to be sold at \$24.95, how many copies should be sold in order to make a profit of \$5000.00?
3. What are the points of intersection of a circle centered at the origin and radius 1, and a line which passes through points $(1, 0)$ and $(0, 1)$.
4. The length of a rectangle is $2''$ longer than its width. If its area is 35 square inches, find its length and width.
5. Which point on the curve $y = \sqrt{x}$ is closest to the point $(1, 0)$?
6. What are the dimensions of the largest (in area) rectangle which can be inscribed in a circle of radius 2?
7. An athletic field with a perimeter of a quarter mile consists of a rectangle with a semicircle at each end. Find the radius r of the circle and the length x of the rectangle that yield the maximum area.



Some thing useful for (5), (6) and (7)

Vertex Formula: $x-$coordinate

The $x-$ coordinate of the vertex of the parabola $y = ax^2 + bx + c$, $a \neq 0$ is given by

$$x = \frac{-b}{2a}$$

Vertex Formula: $y-$coordinate

The $y-$ coordinate of the vertex of the parabola $y = ax^2 + bx + c$, $a \neq 0$ is given by

$$y = \frac{-b^2 + 4ac}{4a}$$

Maximum or Minimum???

Given a parabola $y = ax^2 + bx + c$, $a \neq 0$;

- if $a > 0$, it is a minimum (because it opens up),
- if $a < 0$, it is a maximum (because it opens down)

PROOF: See the details of completing the square for a quadratic function.

Polynomial Functions (§4.6)

Polynomial Function

An equation of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where n is a non-negative integer.
To write a polynomial in **standard form** order the exponents from largest to smallest.

Degree of a Polynomial

The largest exponent in a polynomial.

Leading Coefficient

The coefficient with the variable of the highest degree

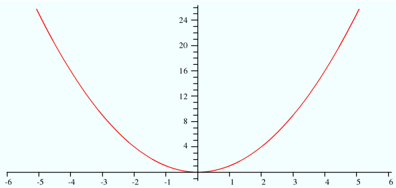
Power Functions

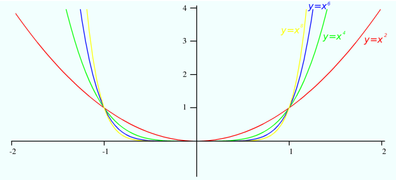
Functions of the form $y = x^n$, where n is any real number.

Q: Which of the basic functions in the front cover of our book are power functions?

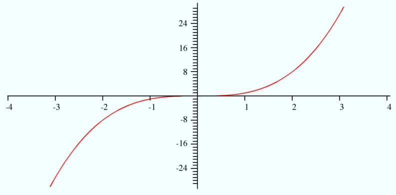
Power Functions: Some important facts

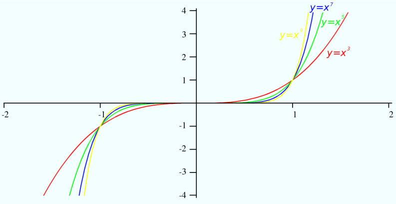
- If n is positive and even: The graph resembles the graph of $y = x^2$



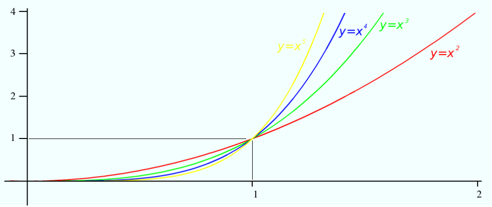


- If n is positive and odd: The graph resembles the graph of $y = x^3$





Power Functions: One more important fact



For two positive integers m and n, with $m < n$, then:

- If $x = 0$ then $x^n = x^m = 0$
- **If** $0 < x < 1$ **then** $0 < x^n < x^m < 1$
- If $x = 1$ then $x^n = x^m = 1$
- **If** $x > 1$ **then** $x^n > x^m > 1$

Properties of Polynomials

- Polynomials will always be continuous smooth curves. (They have not breaks or make sharp/pointy turns.)
- The graph of a polynomial of degree n will have **at most** $n - 1$ turning points.
- As x become very large (positive or negative) the term of highest degree over powers the the rest of the polynomial.

  Thus, the end behavior of a polynomial will match the power function matching the term with the highest degree.
- **excluded regions:** Areas through which the graph cannot pass

Examples: Page 294 - Example 4, Page 299 - Excercise 17 - 24

Sketching Graphs of Polynomials

"**sketch:** Preliminary drawing for later elaboration. Describe roughly or briefly or give the main points or a summary."

Steps for sketching graphs of polynomials

1. Factorize the given polynomial.
2. Find the $x-$intercepts.
3. Find the $y-$intercept.
4. Find the excluded regions. (possibly another table...)
5. Analyze the behavior for large (both positive and negative) values of x.
6. Analyze the behavior near the $x-$intercepts.
7. Note how many turning points the graph can have **at most**.

These steps will provide a *rough* sketch of a graph of a polynomial. We will improve the sketch later...

Sketching Graphs of Polynomials: Examples

Let $y = x^3 - 5x^2 - x + 5$

1. Factorize the given polynomial
   $y = x^3 - 5x^2 - x + 5 = (x+1)(x-1)(x-5)$
2. Find the $x-$intercepts
   Setting $(x-1)(x+1)(x-5) = 0$, gives $x = -1, 1, 5$
3. Find the $y-$intercept
   Setting $x = 0$, we get $y = 5$
4. Find the excluded regions

	Test Point	$(x+1)$	$(x-1)$	$(x-5)$	y
$(-\infty, -1)$	-2	negative	negative	negative	negative
$(-1, 1)$	0	positive	negative	negative	positive
$(1, 5)$	2	positive	positive	negative	negative
$(5, \infty)$	6	positive	positive	positive	positive

Sketching Graphs of Polynomials: Examples

5. Analyze the behavior for large (both positive and negative) values of x

   For large x (positive or negative), the polynomial behaves like the highest power term; in this case x^3

   example: when $x = 1,000$, $x^3 = 1,000,000,000$, $-5x^2 = -5,000,000$ and $-x = -1,000$, then $y = 994,999,005 \approx 1,000,000,000$

6. Analyze the behavior near the $x-$intercepts

   For $x = 1.001$: $y = (1.001 + 1)(1.001 - 1)(1.001 - 5) = (2.001)(0.001)(-3.999) = -0.008001999$.

   For $x = 0.999$: $y = (0.999 + 1)(0.999 - 1)(0.999 - 5) = (1.999)(-0.001)(-4.001) = 0.007997999$.

   On the other hand $(1 + 1)(1.001 - 1)(1 - 5) = (2)(0.001)(-4) = -0.008$,

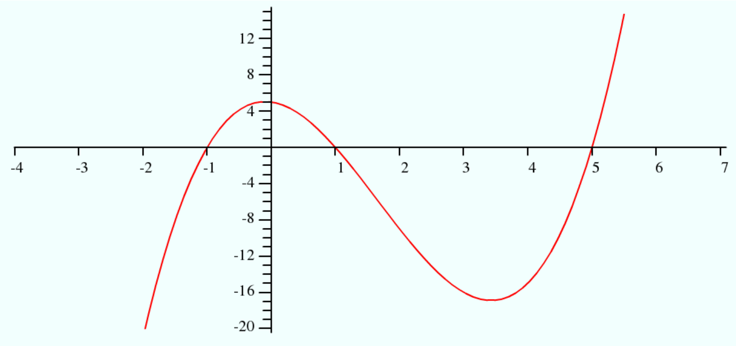
   and $(1 + 1)(0.999 - 1)(1 - 5) = (2)(-0.001)(-4) = 0.008$.

   When x is close to 1, function behaves almost like "linear", and more over, we can approximately find that variation easily...

7. Note how many turning points the graph can have **at most**.

   Since the given polynomial is cubic (degree 3), this can have **at most** 2 turning points.

After all that... the graph



Try these...

1. $y = (x+1)^2(x-1)(x-3)$
2. $y = x^3 - 4x^2 - 5x$
3. $y = x^3 + 3x^2 - 4x - 12$

Rational Functions (§4.7)

Polynomial Function

A function of the form $y = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials of x.

Asymptotes

A line that a function gets infinitely close to but never actually touches or crosses.

- **Vertical:** Occur where the denominator function equals zero. These numbers are excluded from the domain.
- **Horizontal:** Found by comparing the degrees of the polynomials in the numerator and the denominator.
  - Degree of the numerator is greater: NONE
  - Degrees are same: Ratio of the leading coefficient
  - Degree of the denominator is greater: $y = 0$
- **Vertical:** The degree of the numerator is greater by 1. We will find these using polynomial long division.

Sketching Graphs of Rational Functions

Steps for sketching graphs of rational functions

1. Factorize the numerator polynomial and the denominator polynomial.
2. Find the vertical asymptotes .
3. Find the $x-$intercepts.
4. Find the $y-$intercept.
5. Find the horizontal and slant asymptotes .
6. Find the excluded regions. (possibly another table...)
7. Analyze the behavior near the $x-$intercepts and asymptotes.

Sketching Graphs of Rational Functions: Example

Sketch the graph of $y = \dfrac{x^2 + x - 6}{2x^3 + x^2 - 3x}$

Steps for sketching graphs of rational functions

1. Factorize the numerator polynomial and the denominator polynomial.

   $$y = \frac{(x-2)(x+3)}{x(2x+3)(x-1)}$$

2. Find the vertical asymptotes.

   Vertical asymptotes are when the denominator is zero: $x = -3/2, 0, 1$

   These points are not in the domain

3. Find the $x-$intercepts.

   The $x-$intercepts are when the numerator is zero: $x = -3, 2$

4. Find the $y-$intercept.

   The $y-$intercepts are when the $x = 0$; but it is not in the domain

5. Find the horizontal and slant asymptotes .

   Degree of the denominator is greater than the degree of the numerator. Therefore, NO slant asymptotes; horizontal asymptote is $y = 0$.

Sketching Graphs of Rational Functions: Example

6. Find the excluded regions.

   We have to consider all key points: vertical asymptotes ad $x-$intercepts.

   Use the number line to find the regions.

	Test Point	$(x+3)$	$(2x+3)$	x	$(x-1)$	$(x-2)$	y
$(-\infty, -3)$	-6	neg.	neg.	neg.	neg.	neg.	neg.
$(-3, -3/2)$	-2	pos.	neg.	neg.	neg.	neg.	pos.
$(-3/2, 0)$	-1	pos.	pos.	neg.	neg.	neg.	neg.
$(0, 1)$	0.5	pos.	pos.	pos.	neg.	neg.	pos.
$(1, 2)$	1.5	pos.	pos.	pos.	pos.	neg.	neg.
$(2, \infty)$	7	pos.	pos.	pos.	pos.	pos.	pos.

7. Analyze the behavior near the $x-$intercepts and asymptotes....

We can sketch the graph now...