

MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note - Equation of a Plane through Three Given Points

The goal of this note is to give an idea on the equation of planes. In particular, how to obtain the equation of a plane given three points the plane passes through. Please understand the (symbolic) arguments used in and use them to solve the numerical problems. You are NOT required to be able to derive the equations for exams or quizzes. But it will be helpful to know the techniques. You are expected to know how to apply these concepts.

- Find the equation of the plane \mathbb{P} which passes through the points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$.

1. *Method 1: Substitution*

Assume that the the equation of the plane \mathbb{P} is given by $Ax + By + Cz + D = 0$. The idea is to plug-in the coordinates of each point to x , y and z , and then solve for the unknowns A , B , C and D .

$$\text{Point } P \rightsquigarrow Ax_1 + By_1 + Cz_1 + D = 0 \quad (1)$$

$$\text{Point } Q \rightsquigarrow Ax_2 + By_2 + Cz_2 + D = 0 \quad (2)$$

$$\text{Point } R \rightsquigarrow Ax_3 + By_3 + Cz_3 + D = 0 \quad (3)$$

Note that $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3$ and z_3 are known. We need to solve for A, B, C and D .

We have a small problem! We have 3 equations in 4 unknowns! ... However, this is not too bad.

Using equation (1), first solve for D (because it is the easiest):

$$D = -(Ax_1 + By_1 + Cz_1) \quad (4)$$

Substitute this in the other two equations:

$$\text{Equation (2)} \rightsquigarrow A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \quad (5)$$

$$\text{Equation (3)} \rightsquigarrow A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) = 0 \quad (6)$$

Let's eliminate C using these two equations (*the choice is arbitrary ...*):

$$\text{Eq. (5)} \times (z_3 - z_1) \rightsquigarrow A(x_2 - x_1)(z_3 - z_1) + B(y_2 - y_1)(z_3 - z_1) + C(z_2 - z_1)(z_3 - z_1) = 0 \quad (7)$$

$$\text{Eq. (6)} \times (z_2 - z_1) \rightsquigarrow A(x_3 - x_1)(z_2 - z_1) + B(y_3 - y_1)(z_2 - z_1) + C(z_3 - z_1)(z_2 - z_1) = 0 \quad (8)$$

Subtracting equation (7) by equation (8) we get

$$A[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)] + B[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)] = 0 \quad (9)$$

Now we can solve for B in terms of A (*again, the choice is arbitrary ...*):

$$B = - \frac{A[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)]}{[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]} \quad (10)$$

Since we wrote B in terms of A , we *have* to write C in terms of A to have a nice pattern emerging:

In order to eliminate B using (5) and (6):

$$\text{Eq. (5)} \times (y_3 - y_1) \rightsquigarrow A(x_2 - x_1)(y_3 - y_1) + B(y_2 - y_1)(y_3 - y_1) + C(z_2 - z_1)(y_3 - y_1) = 0 \quad (11)$$

$$\text{Eq. (6)} \times (y_2 - y_1) \rightsquigarrow A(x_3 - x_1)(y_2 - y_1) + B(y_3 - y_1)(y_2 - y_1) + C(z_3 - z_1)(y_2 - y_1) = 0 \quad (12)$$

Subtracting equation (11) by equation (12) we get

$$A[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] + C[(z_2 - z_1)(y_3 - y_1) - (z_3 - z_1)(y_2 - y_1)] = 0 \quad (13)$$

Now we can solve for C in terms of A :

$$C = - \frac{A[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]}{[(z_2 - z_1)(y_3 - y_1) - (z_3 - z_1)(y_2 - y_1)]} \quad (14)$$

Absorb the $(-)$ sign to the denominator of (14), and after some reordering,

$$C = \frac{A[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]}{[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]} \quad (15)$$

Note that the denominator of (10) and (15)

Now it's the time for some manipulation ...

We can first plug-in for D , as given by the expression in (4), in the standard equation $Ax + By + Cz + D = 0$:

$$Ax + By + Cz - (Ax_1 + By_1 + Cz_1) = 0 \quad (16)$$

This can be simplified to

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad (17)$$

Now we substitute for B and C using A , as given by (10) and (15)

$$A(x - x_1) - \left(\frac{A[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)]}{[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]} \right) (y - y_1) + \left(\frac{A[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]}{[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]} \right) (z - z_1) = 0 \quad (18)$$

Now we can get everything to a common denominator (*do not worry about the length of the expression ...*)

$$\frac{A[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)](x - x_1) - A[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)](y - y_1) + A[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)](z - z_1)}{[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]} = 0$$

We can factor out $A/[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]$ and it will not change the equation.

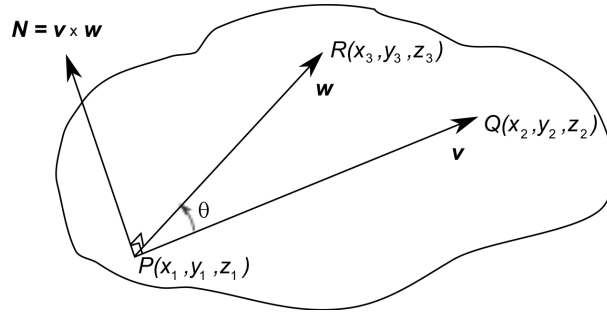
So, the final equation we need will be:

$$[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)](x - x_1) - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)](y - y_1) + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)](z - z_1) = 0$$

We will see the true meaning of this equation in the discussion of the next method

(which happens to be a lot easier as well ...)

2. Method 2: Using the Cross Product



Recall that the equation of a plane can be written easily if we are given the normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, and a point $P = (x_0, y_0, z_0)$ is given. In that case the equation of the plane will be given by the “point – normal” form $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. This is exactly what we are going to use here!

Also recall that the cross product $\mathbf{v} \times \mathbf{w}$ of two vectors \mathbf{v} and \mathbf{w} is perpendicular to the plane of the two vectors \mathbf{v} and \mathbf{w} .

Therefore, set $\mathbf{N} = \mathbf{v} \times \mathbf{w}$, where, $\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$ and $\mathbf{w} = (x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j} + (z_3 - z_1)\mathbf{k}$.

Note that \mathbf{v} is the vector along PQ and \mathbf{w} is the vector along PR .

The normal vector \mathbf{N} passes through the point P , which can be represented by the vector $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$.

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{vmatrix} \quad (1)$$

So,

$$\mathbf{N} = [(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]\mathbf{i} - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)]\mathbf{j} + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\mathbf{k}$$

Therefore, by comparing with the standard notation $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, we immediately see that,

$$\begin{aligned} A &= [(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)] \\ B &= -[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)] \\ C &= [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \end{aligned}$$

At this point, we can appeal to the “point – normal” form and write down the equation of the plane

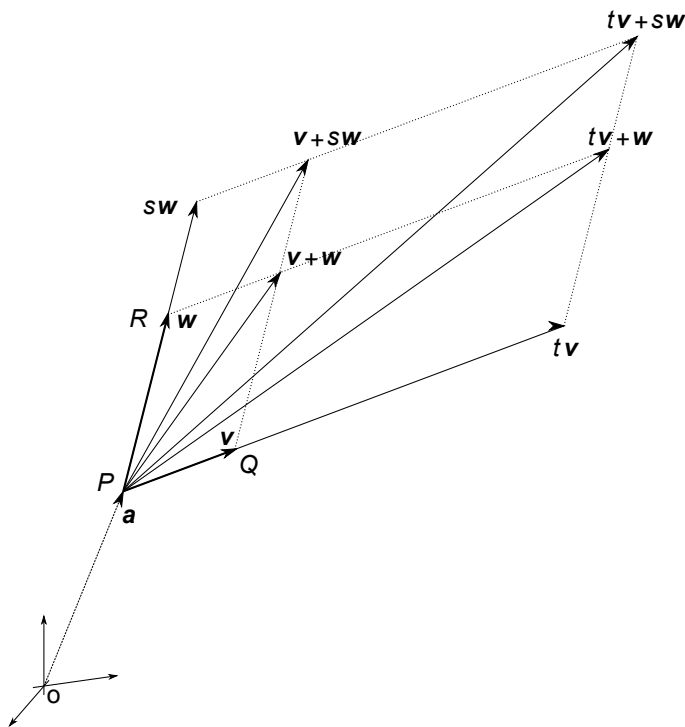
$$[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)](x - x_1) - [(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)](y - y_1) + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)](z - z_1) = 0$$

This is identical to the equation we developed in the previous section!

This is *probably* the easiest method to solve this problem!

The next method is yet another vector method but uses the vector addition only.

3. Method 3: Parametric Equation of a Plane



First note that (by the properties of vector addition using the parallelogram law), for any two scalars t and s , and two vectors \mathbf{v} and \mathbf{w} , the vector given by $t\mathbf{v} + s\mathbf{w}$ lies on the same plane as the plane on which the two vectors \mathbf{v} and \mathbf{w} . Hence the vector quantity $t\mathbf{v} + s\mathbf{w}$ is usually called the “Linear Span” of the two vectors \mathbf{v} and \mathbf{w} .

Using this idea, (similar to what we did in the case of lines) we can write the equation of a plane in the parametric form using the two scalar parameters t and s .

$$\mathbf{a} + t\mathbf{v} + s\mathbf{w}$$

Since \mathbf{v} is the vector parallel to line segment PQ , we get $\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

Since \mathbf{w} is the vector parallel to line segment PR , we get $\mathbf{w} = (x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j} + (z_3 - z_1)\mathbf{k}$.

Since point P is the common point to both PQ and PR , we get $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$.

Put everything together, and we get (after rearranging) the equation of the plane through the points P , Q and R ,

$$[x_1 + t(x_2 - x_1) + s(x_3 - x_1)]\mathbf{i} + [y_1 + t(y_2 - y_1) + s(y_3 - y_1)]\mathbf{j} + [z_1 + t(z_2 - z_1) + s(z_3 - z_1)]\mathbf{k}$$

This gives the parametric form of a line, and it means,

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) + s(x_3 - x_1) \\ y &= y_1 + t(y_2 - y_1) + s(y_3 - y_1) \\ z &= z_1 + t(z_2 - z_1) + s(z_3 - z_1) \end{aligned}$$

We end up with 3 equations in 2 unknowns t and s , you will have to eliminate t and s to obtain the equation of the plane only in x , y and z .

If you do this correctly, you will end up with the same equation as before.

Out of these three methods, the second method is the easiest. The last method gives an interesting way to visualize planes (which can be easily generalized in to things that you may learn in Linear Algebra, for example in MATH 2360).