MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note # 2 – Distance Formulæ

The goal of this note is to give a summary of all the distance formulas in chapter 9.

Prerequisites:

- Given a point P with coordinates (x, y, z) we can associate a vector to the coordinates of the point P as $\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

- Given two points P and Q with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively we can construct a vector from P to Q (or parallel to the line segment PQ, in the direction from P to Q) in the form $\overrightarrow{\boldsymbol{PQ}} = (x_2 - x_1)\,\boldsymbol{i} + (y_2 - y_1)\,\boldsymbol{j} + (z_2 - z_1)\,\boldsymbol{k}$.

- The equation of a line in the *symmetric form* (a.k.a. *standard form*) is given by

  $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

  Then the corresponding *parametric form* is $x = x_0 + at$; $y = y_0 + bt$ and $z = z_0 + ct$; where t is a scalar parameter.

  The corresponding vector form can be written (after some rearranging) as $\boldsymbol{\ell} = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k} + t\,(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})$.

  This can be interpreted as a line passing through the point given by the vector $\boldsymbol{a} = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k}$, and parallel to the vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$.

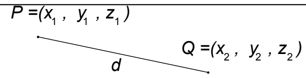
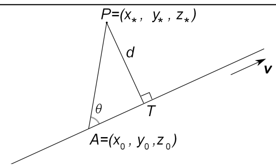
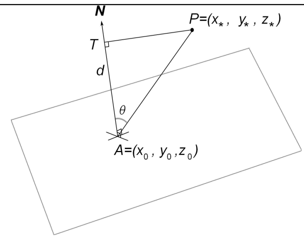
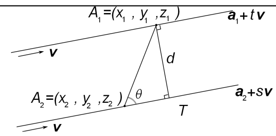
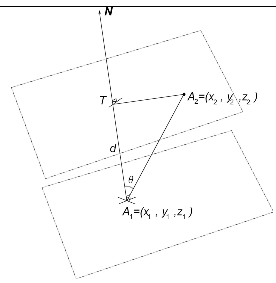
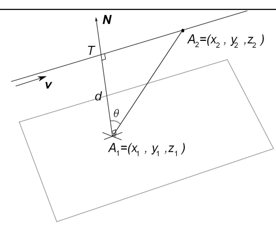
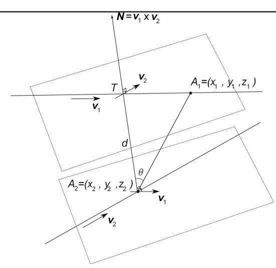
- The equation of a plane can be written in the *point–normal form* as $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

  If we set $D = -\left(Ax_0 + By_0 + Cz_0\right)$, then we can obtain the *standard form* of a plane $Ax + By + Cz + D = 0$.

  The point–normal form implies that the plane passes through the point given by the vector $\boldsymbol{a} = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k}$ and the normal to the plane is parallel to the vector $\boldsymbol{N} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$. i.e. the plane is perpendicular to the vector $\boldsymbol{N}$.

- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the *dot product* is $\boldsymbol{a} \bullet \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\theta$; where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the *cross product* is $\boldsymbol{a} \times \boldsymbol{b} = \left(\|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta\right) \boldsymbol{n}$; where θ is the angle from $\boldsymbol{a}$ to $\boldsymbol{b}$, and $\boldsymbol{n}$ is a unit vector such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{n}$ forms a right-handed system. So, $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta$.

Given Data	Diagram	Vector Equation	Other Formulas
Two points		$d = \left\lVert \overrightarrow{\boldsymbol{PQ}} \right\rVert$	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Point and a line		$d = \dfrac{\left\lVert \overrightarrow{\boldsymbol{AP}} \times \boldsymbol{v} \right\rVert}{\lVert \boldsymbol{v} \rVert}$	$d = PT = AP \sin\theta$
Point and a plane		$d = \dfrac{\left\lvert \overrightarrow{\boldsymbol{AP}} \bullet \boldsymbol{N} \right\rvert}{\lVert \boldsymbol{N} \rVert}$	$d = AT = AP\cos\theta = \dfrac{\lvert Ax_* + By_* + Cz_* + D \rvert}{\sqrt{A^2 + B^2 + C^2}}$
Two parallel lines		$d = \dfrac{\left\lVert \overrightarrow{\boldsymbol{A_1 A_2}} \times \boldsymbol{v} \right\rVert}{\lVert \boldsymbol{v} \rVert}$	$d = A_1T = A_1A_2 \sin\theta$
Two parallel planes		$d = \dfrac{\left\lvert \overrightarrow{\boldsymbol{A_1 A_2}} \bullet \boldsymbol{N} \right\rvert}{\lVert \boldsymbol{N} \rVert}$	$d = A_1T = A_1A_2 \cos\theta$
Parallel line and plane		$d = \dfrac{\left\lvert \overrightarrow{\boldsymbol{A_1 A_2}} \bullet \boldsymbol{N} \right\rvert}{\lVert \boldsymbol{N} \rVert}$	$d = A_1T = A_1A_2 \cos\theta$
Skew lines *		$d = \dfrac{\left\lvert \overrightarrow{\boldsymbol{A_2 A_1}} \bullet (\boldsymbol{v_1} \times \boldsymbol{v_2}) \right\rvert}{\lVert \boldsymbol{v_1} \times \boldsymbol{v_2} \rVert}$	$d = A_2T = A_1A_2 \cos\theta$

* Given two skew lines along $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, we construct two parallel planes with normal $\boldsymbol{v_1} \times \boldsymbol{v_2}$, and one passing through a point A_1 on the first line and the other through a point A_2 on the second line.

In $\mathbb{R}^3$, if a line and a plane are not parallel to one another, then they will surely intersect. So, the distance between a non-parallel line and a plane is always zero!.

How can you check if a line and a plane are parallel to one another?

Suppose the normal of the plane is $\boldsymbol{N}$ and the line is parallel to the vector $\boldsymbol{v}$. Then the line and the plane are parallel if and only if $\boldsymbol{N} \bullet \boldsymbol{v} = 0$.

This means the normal to the plane and the line should be perpendicular to one another. *why and how?*