MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note # 3 – Vector Functions In One Parameter

The goal of this note is to give a summary the concepts in chapter 10.1 *and* 10.2.

- Vector functions of parameter t are of the form $\boldsymbol{F}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$.

  Each component $f_1(t)$, $f_2(t)$ and $f_3(t)$ is functions of t.

- Examples:
  - *a line:* $\boldsymbol{\ell}(t) = (3+4t)\boldsymbol{i} + (2t)\boldsymbol{j} + 9\boldsymbol{k}$
  - *a helix:* $\boldsymbol{h}(t) = (3\cos t)\boldsymbol{i} + (3\sin t)\boldsymbol{j} + 6t\boldsymbol{k}$
  - *a circle (on* $\mathbb{R}^2$*):* $\boldsymbol{c}(\theta) = (2\cos\theta)\boldsymbol{i} + (2\sin\theta)\boldsymbol{j}$

- Addition, subtraction, multiplication by a scalar (constant or by a scalar function of the parameter), dot product, cross product, magnitude, triple products, etc., of vector valued functions are just like for regular vectors.

  You can treat a vector valued function of one parameter pretty much like a regular vector for these operations.

  Example: Consider the two vector valued functions of parameter t defined by $\boldsymbol{F}(t) = (t+1)\boldsymbol{i} + 2\mathrm{e}^t\boldsymbol{j} + (\sin t)\boldsymbol{k}$ and $\boldsymbol{G}(t) = (t-1)\boldsymbol{i} + (\frac{1}{t})\boldsymbol{j} + 6\boldsymbol{k}$, and the scalar valued function of t defined by $h(t) = 2t^2$.

  - *Addition:* Add componentwise

$$(\boldsymbol{F}+\boldsymbol{G})(t) = \boldsymbol{F}(t) + \boldsymbol{G}(t) = (t+1+t-1)\boldsymbol{i} + (2\mathrm{e}^t + \frac{1}{t})\boldsymbol{j} + (\sin t + 6)\boldsymbol{k} = 2t\boldsymbol{i} + (2\mathrm{e}^t + \frac{1}{t})\boldsymbol{j} + (6+\sin t)\boldsymbol{k}$$

  - *Subtraction:* Subtract componentwise

$$(\boldsymbol{F}-\boldsymbol{G})(t) = \boldsymbol{F}(t) - \boldsymbol{G}(t) = [t+1-(t-1)]\boldsymbol{i} + (2\mathrm{e}^t - \frac{1}{t})\boldsymbol{j} + (\sin t - 6)\boldsymbol{k} = 2\boldsymbol{i} + (2\mathrm{e}^t - \frac{1}{t})\boldsymbol{j} + (\sin t - 6)\boldsymbol{k}$$

  - *Multiplication by a scalar:* Multiply each component by the scalar/scalar function

$$(h\boldsymbol{F})(t) = h(t)\boldsymbol{F}(t) = (2t^2)(t+1)\boldsymbol{i} + (2t^2)(2\mathrm{e}^t)\boldsymbol{j} + (2t^2)(\sin t)\boldsymbol{k} = (2t^3+2t^2)\boldsymbol{i} + (4t^2\mathrm{e}^t)\boldsymbol{j} + (2t^2\sin t)\boldsymbol{k}$$

  - *Cross Product:* This results in a vector function of the parameter

$$\begin{aligned}(\boldsymbol{F}\times\boldsymbol{G})(t) = \boldsymbol{F}(t)\times\boldsymbol{G}(t) &= \big((t+1)\boldsymbol{i} + 2\mathrm{e}^t\boldsymbol{j} + (\sin t)\boldsymbol{k}\big) \times \left((t-1)\boldsymbol{i} + \left(\frac{1}{t}\right)\boldsymbol{j} + 6\boldsymbol{k}\right) \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ (t+1) & 2\mathrm{e}^t & (\sin t) \\ (t-1) & (\frac{1}{t}) & 6 \end{vmatrix} \\ &= \left(12\mathrm{e}^t - \left(\frac{1}{t}\right)\sin t\right)\boldsymbol{i} - (6(t+1) - (t-1)\sin t)\boldsymbol{j} + \left(\left(\frac{1}{t}\right)(t+1) - 2(t-1)\mathrm{e}^t\right)\boldsymbol{k}\end{aligned}$$

  - *Dot Product:* This results in a scalar function of the parameter

$$(\boldsymbol{F}\bullet\boldsymbol{G})(t) = \boldsymbol{F}(t)\bullet\boldsymbol{G}(t) = (t+1)(t-1) + (2\mathrm{e}^t)\left(\frac{1}{t}\right) + (\sin t)(6) = t^2 + 6\sin t + 2\mathrm{e}^t/t$$

  - *Magnitude:* This is also a scalar function of the parameter

$$\|\boldsymbol{F}(t)\| = \sqrt{(t+1)^2 + 4\mathrm{e}^{2t} + \sin^2 t}$$

- Limits, continuity, differentiability, differentiation, integration works similar to scalar valued, but now they should work for each component:

  Examples:

  - ***Limits:*** Find limits of each component.

    Given $\boldsymbol{F}(t) = (t+1)\boldsymbol{i} + \left(\frac{\sin t}{t}\right)\boldsymbol{j} + (\mathrm{e}^t)\boldsymbol{k}$, find $\lim_{t\to 0} \boldsymbol{F}(t)$:

$$\begin{aligned}\lim_{t\to 0} \boldsymbol{F}(t) &= \left(\lim_{t\to 0} f_1(t)\right)\boldsymbol{i} + \left(\lim_{t\to 0} f_2(t)\right)\boldsymbol{j} + \left(\lim_{t\to 0} f_3(t)\right)\boldsymbol{k} \\ &= \left(\lim_{t\to 0} t+1\right)\boldsymbol{i} + \left(\lim_{t\to 0} \frac{\sin t}{t}\right)\boldsymbol{j} + \left(\lim_{t\to 0} \mathrm{e}^t\right)\boldsymbol{k} \\ &= (1)\,\boldsymbol{i} + (1)\,\boldsymbol{j} + (1)\,\boldsymbol{k}\end{aligned}$$

  - ***Continuity:*** See if each component is continuous

    Investigate the continuity of $\boldsymbol{F}(t) = (t+1)\boldsymbol{i} + \left(\frac{1}{t}\right)\boldsymbol{j} + (\tan t)\boldsymbol{k}$

    The $\boldsymbol{i}$ component is continuous at any real number

    The $\boldsymbol{j}$ component is continuous at any real number except at $t = 0$

    The $\boldsymbol{k}$ component is continuous at any real number except when t is an odd multiple of $\pi/2$

    Therefore, the vector valued function $\boldsymbol{F}$ is continuous at every real number except 0 or an odd multiple of $\pi/2$

  - ***Differentiability:*** See if/when each component is differentiable

  - ***Differentiation:*** Differentiate componentwise

    Differentiate $\boldsymbol{F}(t) = (t^4 + \mathrm{e}^{-t})\boldsymbol{i} + \left(\cos t^2\right)\boldsymbol{j} + (\tan t)\boldsymbol{k}$, w.r.t. t

$$\begin{aligned}\boldsymbol{F}'(t) &= f_1'(t)\boldsymbol{i} + f_2'(t)\boldsymbol{j} + f_3'(t)\boldsymbol{k} \\ &= \left(4t^3 - \mathrm{e}^{-t}\right)\boldsymbol{i} + \left(-2t\sin t^2\right)\boldsymbol{j} + \left(\sec^2 t\right)\boldsymbol{k}\end{aligned}$$

  - ***Integration:*** Intrgrate componentwise

    Find the indefinite integral of $\boldsymbol{F}(t) = (t^4 + \mathrm{e}^{-t})\boldsymbol{i} + (\cos t)\,\boldsymbol{j} + \left(\frac{1}{t}\right)\boldsymbol{k}$, w.r.t. t

$$\begin{aligned}\int \boldsymbol{F}(t)\,\mathrm{d}t &= \left(\int f_1(t)\,\mathrm{d}t\right)\boldsymbol{i} + \left(\int f_2(t)\,\mathrm{d}t\right)\boldsymbol{j} + \left(\int f_3(t)\,\mathrm{d}t\right)\boldsymbol{k} \\ &= \left(\int t^4 + \mathrm{e}^{-t}\,\mathrm{d}t\right)\boldsymbol{i} + \left(\int \cos t\,\mathrm{d}t\right)\boldsymbol{j} + \left(\int \frac{1}{t}\,\mathrm{d}t\right)\boldsymbol{k} \\ &= \left(\frac{t^5}{5} - \mathrm{e}^{-t} + c_1\right)\boldsymbol{i} + (\sin t + c_2)\,\boldsymbol{j} + (\ln|t| + c_3)\,\boldsymbol{k}\end{aligned}$$

    Find the definite integral of $\boldsymbol{F}(t) = t\boldsymbol{i} + (\sin \pi t)\,\boldsymbol{j} + (\mathrm{e}^t)\,\boldsymbol{k}$, from $t = 1$ to $t = 2$.

$$\begin{aligned}\int_1^2 \boldsymbol{F}(t)\,\mathrm{d}t &= \left(\int_1^2 f_1(t)\,\mathrm{d}t\right)\boldsymbol{i} + \left(\int_1^2 f_2(t)\,\mathrm{d}t\right)\boldsymbol{j} + \left(\int_1^2 f_3(t)\,\mathrm{d}t\right)\boldsymbol{k} \\ &= \left(\int_1^2 t\,\mathrm{d}t\right)\boldsymbol{i} + \left(\int_1^2 \sin \pi t\,\mathrm{d}t\right)\boldsymbol{j} + \left(\int_1^2 \mathrm{e}^t\,\mathrm{d}t\right)\boldsymbol{k} \\ &= \left[\frac{t^2}{2}\right]_1^2 \boldsymbol{i} + \left[\frac{\cos \pi t}{\pi}\right]_1^2 \boldsymbol{j} + \left[\mathrm{e}^t\right]_1^2 \boldsymbol{k} \\ &= \left(\frac{2^2 - 1^2}{2}\right)\boldsymbol{i} + \left(\frac{\cos 2\pi - \cos \pi}{\pi}\right)\boldsymbol{j} + \left(\mathrm{e}^2 - \mathrm{e}^1\right)\boldsymbol{k} = \left(\frac{3}{2}\right)\boldsymbol{i} + \left(\frac{2}{\pi}\right)\boldsymbol{j} + \left(\mathrm{e}^2 - \mathrm{e}^1\right)\boldsymbol{k}\end{aligned}$$

- *Properties:* Let $\boldsymbol{F}(t)$ and $\boldsymbol{G}(t)$ be vector valued functions and $h(t)$ be a scalar valued function of t. Also let a and b be scalar constants.
  - ***Rules of Limits***
    - $\lim_{t\to t_0} (\boldsymbol{F}(t) + \boldsymbol{G}(t)) = \lim_{t\to t_0} \boldsymbol{F}(t) + \lim_{t\to t_0} \boldsymbol{G}(t)$
    - $\lim_{t\to t_0} (\boldsymbol{F}(t) - \boldsymbol{G}(t)) = \lim_{t\to t_0} \boldsymbol{F}(t) - \lim_{t\to t_0} \boldsymbol{G}(t)$
    - $\lim_{t\to t_0} (h(t)\boldsymbol{F}(t)) = \left(\lim_{t\to t_0} h(t)\right)\left(\lim_{t\to t_0} \boldsymbol{F}(t)\right)$
    - $\lim_{t\to t_0} (\boldsymbol{F}(t) \bullet \boldsymbol{G}(t)) = \left(\lim_{t\to t_0} \boldsymbol{F}(t)\right) \bullet \left(\lim_{t\to t_0} \boldsymbol{G}(t)\right)$
    - $\lim_{t\to t_0} (\boldsymbol{F}(t) \times \boldsymbol{G}(t)) = \left(\lim_{t\to t_0} \boldsymbol{F}(t)\right) \times \left(\lim_{t\to t_0} \boldsymbol{G}(t)\right)$
  - ***Rules of Differentiation***
    - $(a\boldsymbol{F} + b\boldsymbol{G})'(t) = a\boldsymbol{F}'(t) + b\boldsymbol{G}'(t)$
    - $(h\boldsymbol{F})'(t) = h'(t)\boldsymbol{F}(t) + h(t)\boldsymbol{F}'(t)$
    - $(\boldsymbol{F} \bullet \boldsymbol{G})'(t) = \big(\boldsymbol{F}' \bullet \boldsymbol{G}\big)(t) + \big(\boldsymbol{F} \bullet \boldsymbol{G}'\big)(t)$
    - $(\boldsymbol{F} \times \boldsymbol{G})'(t) = \big(\boldsymbol{F}' \times \boldsymbol{G}\big)(t) + \big(\boldsymbol{F} \times \boldsymbol{G}'\big)(t)$
    - $(\boldsymbol{F}(h(t)))' = h'(t)\boldsymbol{F}'(h(t))$ *(Chain Rule)*

- ***Smooth Vector Valued Function:*** The vector valued function $\boldsymbol{F}$ is continuous and $\boldsymbol{F}'(t) \neq 0$.
  Piecewise smooth if there is a finite number of subintervals on which $\boldsymbol{F}$ is smooth.

- In general $\boldsymbol{F}'(t)$ gives the *tangent* to the vector valued function $\boldsymbol{F}$.

- If the vector valued function $\boldsymbol{R}(t)$ describes the variation of the position of a "particle" with time t then
  - $\boldsymbol{R}'(t)$ is the *velocity* of the particle at time t
  - $\boldsymbol{R}'(t)/\|\boldsymbol{R}'(t)\|$ is the *direction of motion* of the particle at time t
  - $\boldsymbol{R}''(t)$ is the *acceleration* of the particle at time t

- If $\|\boldsymbol{F}(t)\|$ is constant, then $\boldsymbol{F}'(t)$ is orthogonal to $\boldsymbol{F}$. *we can prove this easily.*

  Suppose $\|\boldsymbol{F}(t)\| = k$, constant.
  Then, $\|\boldsymbol{F}(t)\|^2 = k^2$.
  We had a property of the magnitude that $\|\boldsymbol{F}(t)\|^2 = \boldsymbol{F}(t) \bullet \boldsymbol{F}(t)$.
  Hence, $\boldsymbol{F}(t) \bullet \boldsymbol{F}(t) = k^2$, still it is constant.
  So, differentiating the dot product $\boldsymbol{F}'(t) \bullet \boldsymbol{F}(t) + \boldsymbol{F}(t) \bullet \boldsymbol{F}'(t) = 0$.
  Then we see that $\boldsymbol{F}(t) \bullet \boldsymbol{F}'(t) = 0$. i.e. they are orthogonal.