MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note # 4 – Two Problems

Two similar looking problems which are actually quite different.

For example: Two bees fly on a path modeled using one parameter t with $t \geq 0$, by the two vector valued functions $\boldsymbol{b}_1(t) = (t+1)\cos(2\pi t)\boldsymbol{i} + (t+1)\sin(2\pi t)\boldsymbol{j} + 2(t+1)\boldsymbol{k}$ and $\boldsymbol{b}_2(t) = 2(t-1)\cos(2\pi t)\boldsymbol{i} + 2(t-1)\sin(2\pi t)\boldsymbol{j} + 2t\boldsymbol{k}$.

1. Find the minimum distance between the two bees
2. Find the points at which the paths traced by their flight intersect

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Hints for (1):

- Use the same parameter t for both equations (think *"time"*)
- The *distance vector* $\boldsymbol{d}(t)$ between the two bees at any time t is $\boldsymbol{d}(t) = \boldsymbol{b}_2(t) - \boldsymbol{b}_1(t)$
- The *distance* between the two bees will be given by $\|\boldsymbol{d}(t)\|$
- This is, in fact, a optimization problem in calculus

Hints for (2):

- Use two names for the parameters: "t" for the first bee and "s" for the second bee (*DO NOT think "time"!*)
  $\boldsymbol{b}_1(t) = (t+1)\cos(2\pi t)\boldsymbol{i} + (t+1)\sin(2\pi t)\boldsymbol{j} + 2(t+1)\boldsymbol{k}$ and $\boldsymbol{b}_2(t) = 2(s-1)\cos(2\pi t)\boldsymbol{i} + 2(s-1)\sin(2\pi t)\boldsymbol{j} + 2s\boldsymbol{k}$
- Set the components equal to one another; just like what we did to find the points of intersection of lines

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Solution for (1):

$$\begin{aligned}
\boldsymbol{d}(t) &= [2(t-1)-(t+1)]\cos(2\pi t)\boldsymbol{i} + [2(t-1)-(t+1)]\sin(2\pi t)\boldsymbol{j} + [2t-2(t+1)]\,\boldsymbol{k} \\
&= (t-3)\cos(2\pi t)\boldsymbol{i} + (t-3)\sin(2\pi t)\boldsymbol{j} - 2\boldsymbol{k} \\
\|\boldsymbol{d}(t)\| &= \|(t-3)\cos(2\pi t)\boldsymbol{i} + (t-3)\sin(2\pi t)\boldsymbol{j} - 2\boldsymbol{k}\| \\
&= \sqrt{[(t-3)\cos(2\pi t)]^2 + [(t-3)\sin(2\pi t)]^2 + (-2)^2} \\
&= \sqrt{(t-3)^2\left[\cos^2(2\pi t) + \sin^2(2\pi t)\right] + 4} \\
&= \sqrt{t^2 - 6t + 9 + 4} \\
&= \sqrt{t^2 - 6t + 13}
\end{aligned}$$

To find the extreme points, take the derivative of $\|\boldsymbol{d}(t)\|$ and set it to zero ...

$$\begin{aligned}
\frac{\mathrm{d}\,\|\boldsymbol{d}(t)\|}{\mathrm{d}t} &= \frac{\mathrm{d}\sqrt{t^2-6t+13}}{\mathrm{d}t} \\
&= \frac{2t-6}{2\sqrt{t^2-6t+13}}
\end{aligned}$$

Setting this to zero, we get $2t - 6 = 0$, so we get $t = 3$.

Note that the denominator of $\mathrm{d}\,\|\boldsymbol{d}(t)\|/\mathrm{d}t$ is always positive and the numerator is negative for $t < 3$ and for $t > 3$ it is positive. So $t = 0$ gives a minimum. At $t = 3$, the distance between the bees $\|\boldsymbol{d}(t)\| = \sqrt{3^2 - (6)(3) + 13} = \sqrt{9 - 18 + 13} = \sqrt{4} = 2$. Since we can only have $t \geq 0$, $t = 0$ is the lower end point and the t we found is $3 > 0$, we are done! Minimum distance is 2, when $t = 3$. Clearly, the two bees never meet at the same point.

Solution for (2):

Following the hint, Use two names for the parameters t for the first bee and s for the second bee.
Path traced by the first bee's flight: $\boldsymbol{b}_1(t) = (t+1)\cos(2\pi t)\boldsymbol{i} + (t+1)\sin(2\pi t)\boldsymbol{j} + 2(t+1)\boldsymbol{k}$
Path traced by the second bee's flight: $\boldsymbol{b}_2(t) = 2(s-1)\cos(2\pi t)\boldsymbol{i} + 2(s-1)\sin(2\pi t)\boldsymbol{j} + 2s\boldsymbol{k}$
Set the corresponding components equal to one another:

$$\begin{aligned}
\boldsymbol{i} &\quad\rightsquigarrow\quad (t+1)\cos(2\pi t) = 2(s-1)\cos(2\pi t) && (1)\\
\boldsymbol{j} &\quad\rightsquigarrow\quad (t+1)\sin(2\pi t) = 2(s-1)\sin(2\pi t) && (2)\\
\boldsymbol{k} &\quad\rightsquigarrow\quad 2(t+1) = 2s && (3)
\end{aligned}$$

Equation (1) and (2) gives the same information: $t+1 = 2(s-1)$, which can be simplified to

$$t + 3 = 2s \tag{4}$$

So, from (3) and (4), we get

$$2t + 2 = t + 3 \tag{5}$$

So, $t = 1$. hence from (4) we get $s = (1+3)/2 = 2$.
Now you should go back and check if these numbers satisfy the equations (1), (2) and (3).
So, we can conclude that the paths traced by the two bees flight intersect at
$\boldsymbol{b}_1(1) = (1+1)\cos(2\pi 1)\boldsymbol{i} + (1+1)\sin(2\pi 1)\boldsymbol{j} + 2(1+1)\boldsymbol{k} = 2\boldsymbol{i} + 0\boldsymbol{j} + 4\boldsymbol{k}$. Which is actually the vector notation for the point with coordinates $(2, 0, 4)$.
If you evaluated $\boldsymbol{b}_2(2)$, it would be the same point.

This means that the second bee (at time 2) passes through the same point (with coordinates $(2, 0, 4)$) which the first bee passed through at a different time (time 1, in this case).

Moral of the story ...

- The two problems, though looks quite the same, are in fact, very different
- Parameters can be interpreted in different ways to mean different things - depending on the context.
- Proper interpretation of the problem is crucial.
