# MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note # 5 – Equations of Motion

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*A summary of §10.3 and a little bit more*

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**Projectiles: *An Example***

Suppose that a baseball is thrown from the roof of a building on a flat ground. Suppose that when the ball leaves the hand of the thrower, it is 60 feet above the ground at the base of the building. Also suppose that the ball is thrown at a velocity $24\boldsymbol{i} + 10\boldsymbol{j}$ ft/s; where $\boldsymbol{i}$ gives the horizontal direction and $\boldsymbol{j}$ gives the vertical direction. Given that the gravitational acceleration is 32 ft/s$^2$, find the equation of motion.

*Solution*

Suppose $\boldsymbol{R}(t)$ gives the position of the ball at time $t$. Then, we know that, velocity, $\boldsymbol{v}(t) = \boldsymbol{R}'(t)$ and acceleration, $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{R}''(t)$.

Also, from the given data, $\boldsymbol{a}(t) = 0\boldsymbol{i} - 32\boldsymbol{j}$, since the gravitational acceleration is constant and directed vertically down, initial velocity $\boldsymbol{v}(0) = 24\boldsymbol{i} + 10\boldsymbol{j}$ and initial position $\boldsymbol{R}(0) = 0\boldsymbol{i} + 60\boldsymbol{j}$.

$\boldsymbol{v}(t) - \boldsymbol{v}(0) = \displaystyle\int_0^t \boldsymbol{a}(\tau)\,\mathrm{d}\tau$

So, $\boldsymbol{v}(t) - (24\boldsymbol{i} + 10\boldsymbol{j}) = \displaystyle\int_0^t (0\boldsymbol{i} - 32\boldsymbol{j})\ \mathrm{d}\tau = \left(\int_0^t 0\ \mathrm{d}\tau\right)\boldsymbol{i} + \left(\int_0^t -32\ \mathrm{d}\tau\right)\boldsymbol{j} = [0\tau]_0^t\,\boldsymbol{i} + [-32\tau]_0^t\,\boldsymbol{j} = 0\boldsymbol{i} - 32t\boldsymbol{j}$.

Therefore, $\boldsymbol{v}(t) = 24\boldsymbol{i} + (10 - 32t)\boldsymbol{j}$.

Then, $\boldsymbol{R}(t) - \boldsymbol{R}(0) = \displaystyle\int_0^t \boldsymbol{v}(\tau)\,\mathrm{d}\tau$

So, $\boldsymbol{R}(t) - (0\boldsymbol{i} + 60\boldsymbol{j}) = \displaystyle\int_0^t (24\boldsymbol{i} + (10 - 32t)\boldsymbol{j})\ \mathrm{d}\tau = \left(\int_0^t 24\ \mathrm{d}\tau\right)\boldsymbol{i} + \left(\int_0^t (10 - 32t)\ \mathrm{d}\tau\right)\boldsymbol{j} = [24\tau]_0^t\,\boldsymbol{i} + \left[10\tau - 16\tau^2\right]_0^t\boldsymbol{j}$

$$= 24t\boldsymbol{i} + (10t - 16t^2)\boldsymbol{j}.$$

Therefore, $\boldsymbol{R}(t) = 24t\boldsymbol{i} + (60 + 10t - 16t^2)\boldsymbol{j}$. Gives the position of the ball at time $t$, before it hits the ground.

$\boldsymbol{R}(0) = 0\boldsymbol{i} + 60\boldsymbol{j}$ $\quad\surd$

$\boldsymbol{v}(t) = \boldsymbol{R}'(t) = 24\boldsymbol{i} + (10 - 32t)\boldsymbol{j} \implies \boldsymbol{v}(0) = 24\boldsymbol{i} + 10\boldsymbol{j}$ $\quad\surd$

$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{R}''(t) = 0\boldsymbol{i} - 32\boldsymbol{j} \implies \boldsymbol{a}(0) = 0\boldsymbol{i} - 32\boldsymbol{j}$ $\quad\surd$

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**Unit Vectors in Polar Coordinates:**

- Unit vector along the radial ($r$) direction $\boldsymbol{u}_r = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$
- Unit vector along the tangential ($\theta$) direction $\boldsymbol{u}_\theta = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}$

Unlike the unit vectors along the Cartesian unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, these polar unit vectors $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ vary with the angle variable $\theta$.

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- $\dfrac{\mathrm{d}\boldsymbol{u}_r}{\mathrm{d}\theta} = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j} = \boldsymbol{u}_\theta$
- $\dfrac{\mathrm{d}\boldsymbol{u}_\theta}{\mathrm{d}\theta} = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j} = -\boldsymbol{u}_r$

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***General Equations of Motion Using Polar Coordinates***

*Recall that, if $(r,\theta)$ and $(x,y)$ are respectively the polar and rectangular Cartesian coordinates of a point. Then, $x = r\cos\theta$ and $y = r\sin\theta$.*

Suppose that the motion of a particle is described using the time parameter $t$ in polar coordinates: $r = r(t)$ and $\theta = \theta(t)$.

Then, the position vector of the particle is: $\boldsymbol{R}(t) = r(t)\cos(\theta(t))\boldsymbol{i} + r(t)\sin(\theta(t))\boldsymbol{j}$.

For brevity let us suppress the $t$ and write the same as: $\boldsymbol{R} = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j}$.

$$\begin{aligned}
\text{Velocity, } \boldsymbol{v}(t) &= \boldsymbol{R}' = (r'\cos\theta - r\theta'\sin\theta)\,\boldsymbol{i} + (r'\sin\theta + r\theta'\cos\theta)\,\boldsymbol{j} \\
&= r'(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) + r\theta'(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) \\
\boldsymbol{v}(t) &= r'\boldsymbol{u}_r + r\theta'\boldsymbol{u}_\theta
\end{aligned}$$

Similarly you can show that acceleration $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{R}''(t) = \left[r'' - (r)(\theta')^2\right]\boldsymbol{u}_r + \left[(r)(\theta'') + 2(r')(\theta')\right]\boldsymbol{u}_\theta$

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*Example:*

Find the components of velocity and acceleration components along $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ when $r = t^2$ and $\theta = t$.
Note that this describes the motion $\boldsymbol{R}(t) = t^2\cos(t)\boldsymbol{i} + t^2\sin(t)\boldsymbol{j}$
$r'(t) = 2t$, $r''(t) = 2$, $\theta'(t) = 1$ and $\theta''(t) = 0$.

So,
$\boldsymbol{v}(t) = 2t\boldsymbol{u}_r + (t^2)(1)\boldsymbol{u}_\theta = 2t\boldsymbol{u}_r + t^2\boldsymbol{u}_\theta$,
$\boldsymbol{a}(t) = \left[2 - (t^2)(1)^2\right]\boldsymbol{u}_r + \left[(t^2)(0) + 2(2t)(1)\right]\boldsymbol{u}_\theta = \left(2 - t^2\right)\boldsymbol{u}_r + (4t)\,\boldsymbol{u}_\theta$

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