MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

Supplementary Note # 6 – Theory of Curves

A summary of §10.4 and a little bit more

Arc length of a curve between two points on a curve

- *For a scalar function; $y = f(x)$ in $\mathbb{R}^2$:* $$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x$$
- *For a parametric function; $x = f_1(t); y = f_2(t)$ in $\mathbb{R}^2$:* $$s = \int_{t_1}^{t_2} \sqrt{[f_1'(t)]^2 + [f_2'(t)]^2}\,\mathrm{d}t$$
- *For a parametric function; $x = f_1(t);\ y = f_2(t);\ z = f_3(t)$ in $\mathbb{R}^3$:* $$s = \int_{t_1}^{t_2} \sqrt{[f_1'(t)]^2 + [f_2'(t)]^2 + [f_3'(t)]^2}\,\mathrm{d}t$$

Arc length function of a curve starting at $t = t_0$

- *For a vector valued fun.; $\boldsymbol{R}(t) = r_1(t)\boldsymbol{i} + r_2(t)\boldsymbol{j}$ in $\mathbb{R}^2$:* $$s(t) = \int_{t_0}^{t} \sqrt{[r_1'(\tau)]^2 + [r_2'(\tau)]^2}\,\mathrm{d}\tau = \int_{t_0}^{t} \|\boldsymbol{R}'(\tau)\|\,\mathrm{d}\tau$$
- *For a vec. val. fun.; $\boldsymbol{R}(t) = r_1(t)\boldsymbol{i} + r_2(t)\boldsymbol{j} + r_3(t)\boldsymbol{k}$ in $\mathbb{R}^3$:* $$s(t) = \int_{t_0}^{t} \sqrt{[r_1'(\tau)]^2 + [r_2'(\tau)]^2 + [r_3'(\tau)]^2}\,\mathrm{d}t = \int_{t_0}^{t} \|\boldsymbol{R}'(\tau)\|\,\mathrm{d}\tau$$

If the vector function $\boldsymbol{R}(t)$ defines the motion of a particle, then the arc length function $s(T)$ defines the distance traveled by the particle from the starting time $t = t_0$, up until time $t = T$. So, $\frac{\mathrm{d}s}{\mathrm{d}t}$ gives the rate of change of distance, *the speed*, $\|\boldsymbol{R}'(\tau)\|$.

We can parameterize a curve using the arc length. Then the equation will tell us how the vector valued function changes "after traveling a particular distance on the curve"; instead of telling us how the vector valued function changes at a particular "time".

Example 1: Parameterize the helix $\boldsymbol{R}(t) = 3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + 4t\boldsymbol{k}; \quad t \geq 0$ using the arc length.

Arc length function $s(t) = \int_0^t \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (4)^2}\,\mathrm{d}t = \int_0^t \sqrt{9(\cos^2 t + \sin^2 t) + 16}\,\mathrm{d}t = \int_0^t \sqrt{9 + 16}\,\mathrm{d}t = 5t$

So, $t = s/5$. Hence, $\boldsymbol{R}(t) = \tilde{\boldsymbol{R}}(s) = 3\sin(s/5)\boldsymbol{i} + 3\cos(s/5)\boldsymbol{j} + (4s/5)\boldsymbol{k}; \quad s \geq 0$

Let $\boldsymbol{R}(t) = r_1(t)\boldsymbol{i} + r_2(t)\boldsymbol{j} + r_3(t)\boldsymbol{k}$ be the vector equation of a curve in $\mathbb{R}^3$. If $\boldsymbol{R}(t)$ is a smooth curve (i.e. differentiable in t and $\boldsymbol{R}'(t) \neq 0$) then we can define the following.

1. *Unit Tangent* $= \boldsymbol{T}(t) = \dfrac{\boldsymbol{R}'(t)}{\|\boldsymbol{R}'(t)\|}$
2. *(Principal) Unit Normal*$= \boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$
3. *Unit Bi-normal*$= \boldsymbol{B}(t) = \dfrac{\boldsymbol{T}(t) \times \boldsymbol{N}(t)}{\|\boldsymbol{T}(t) \times \boldsymbol{N}(t)\|}$

Obviously, $\boldsymbol{T}(t)$ is tangential to the curve described by $\boldsymbol{R}(t)$. Note that $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ are orthogonal to one another. $\boldsymbol{N}(t)$ is tangential to $\boldsymbol{T}(t)/\|\boldsymbol{T}(t)\|$. Since $\boldsymbol{T}(t)/\|\boldsymbol{T}(t)\|$ has a constant magnitude of 1, $\boldsymbol{N}(t)$ is also orthogonal to $\boldsymbol{T}(t)$. Clearly $\boldsymbol{B}(t)$ is orthogonal to $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ since it is defined using the cross product between them.

If we use the arc length parameterization, we can make some nice (and useful) observations.

1. $\boldsymbol{T} = \dfrac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s}$
2. *Curvature* $= \kappa = \left\| \dfrac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s} \right\| = \dfrac{\|\boldsymbol{R}' \times \boldsymbol{R}''\|}{\|\boldsymbol{R}'\|^3}$
3. $\boldsymbol{N} = \dfrac{1}{\kappa}\dfrac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}$

Example 1 (cont'd ...): Compute the Tangent, Normal and Curvatures for the helix given before.

Using the time parameterization

$$
\begin{aligned}
\boldsymbol{R}(t) &= 3\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j} + 4t\boldsymbol{k} \\
\boldsymbol{R}'(t) &= 3\cos t\boldsymbol{i} - 3\sin t\boldsymbol{j} + 4\boldsymbol{k} \\
\|\boldsymbol{R}'(t)\| &= 5 \\
\star \quad \boldsymbol{T}(t) &= \frac{\boldsymbol{R}'(t)}{\|\boldsymbol{R}'(t)\|} = \frac{3}{5}\cos t\boldsymbol{i} - \frac{3}{5}\sin t\boldsymbol{j} + \frac{4}{5}\boldsymbol{k} \\
\boldsymbol{T}'(t) &= -\frac{3}{5}\sin t\boldsymbol{i} - \frac{3}{5}\cos t\boldsymbol{j} + 0\boldsymbol{k} \\
\|\boldsymbol{T}'(t)\| &= \sqrt{\left[-\frac{3}{5}\sin t\right]^2 + \left[-\frac{3}{5}\cos t\right]^2 + 0^2} = \frac{3}{5} \\
\star \quad \boldsymbol{N}(t) &= \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|} = -\cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 0\boldsymbol{k} \\
\boldsymbol{R}''(t) &= -3\sin t\boldsymbol{i} - 3\cos t\boldsymbol{j} + 0\boldsymbol{k} \\
\boldsymbol{R}'(t) \times \boldsymbol{R}''(t) &= 12\cos t\boldsymbol{i} - 12\sin t\boldsymbol{j} - 9\boldsymbol{k} \\
\|\boldsymbol{R}'(t) \times \boldsymbol{R}''(t)\| &= 15 \\
\kappa(t) &= \frac{\|\boldsymbol{R}'(t) \times \boldsymbol{R}''(t)\|}{\|\boldsymbol{R}'(t)\|^3} = \frac{15}{5^3} \\
\star \quad \kappa(t) &= \frac{3}{25}
\end{aligned}
$$

Using the arc length parameterization

$$
\begin{aligned}
s &= 5t \\
\tilde{\boldsymbol{R}}(s) &= 3\sin(s/5)\boldsymbol{i} + 3\cos(s/5)\boldsymbol{j} + (4s/5)\boldsymbol{k} \\
* \quad \tilde{\boldsymbol{T}}(s) &= \frac{\mathrm{d}\tilde{\boldsymbol{R}}(s)}{\mathrm{d}s} = \frac{3}{5}\cos(s/5)\boldsymbol{i} - \frac{3}{5}\sin(s/5)\boldsymbol{j} + \frac{4}{5}\boldsymbol{k} \\
\star \quad \boldsymbol{T}(t) &= \left.\frac{\mathrm{d}\tilde{\boldsymbol{R}}(s)}{\mathrm{d}s}\right|_{s=5t} = \frac{3}{5}\cos t\boldsymbol{i} - \frac{3}{5}\sin t\boldsymbol{j} + \frac{4}{5}\boldsymbol{k} \\
\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s} &= -\frac{3}{25}\sin(s/5)\boldsymbol{i} - \frac{3}{25}\cos(s/5)\boldsymbol{j} + 0\boldsymbol{k} \\
\tilde{\kappa}(s) &= \left\|\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s}\right\| = \sqrt{\left[-\frac{3}{25}\sin t\right]^2 + \left[-\frac{3}{25}\cos t\right]^2 + 0^2} \\
* \quad \tilde{\kappa}(s) &= \left\|\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s}\right\| = \frac{3}{25} \\
\star \quad \kappa(t) &= \left.\left\|\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s}\right\|\right|_{s=5t} = \frac{3}{25} \\
* \quad \tilde{\boldsymbol{N}}(s) &= \frac{1}{\tilde{\kappa}(s)}\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s} = -\sin(s/5)\boldsymbol{i} - \cos(s/5)\boldsymbol{j} + 0\boldsymbol{k} \\
\star \quad \boldsymbol{N}(t) &= \left.\frac{1}{\tilde{\kappa}(s)}\frac{\mathrm{d}\tilde{\boldsymbol{T}}(s)}{\mathrm{d}s}\right|_{s=5t} = -\sin(t)\boldsymbol{i} - \cos(t)\boldsymbol{j} + 0\boldsymbol{k}
\end{aligned}
$$

So, $\boldsymbol{T}(t) = \frac{3}{5}\cos t\boldsymbol{i} - \frac{3}{5}\sin t\boldsymbol{j} + \frac{4}{5}\boldsymbol{k}$; $\boldsymbol{N}(t) = -\cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 0\boldsymbol{k}$; $\kappa = \frac{3}{25}$

I prefer to use a tilde $\tilde{\ }$ to denote arc length parameterized quantities; that is just a convention of my own and it is NOT a standard practice.

The mathematical meaning of *curvature*, κ, is similar to its English meaning: it measures how much a curve *"curves"*! As you would guess, the curvature of a straight line is 0; and for a circle with radius r, the curvature is $1/r$. *(verify)*
