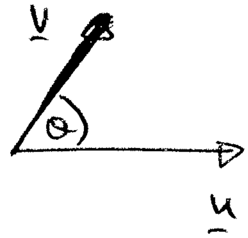
# Major topics in CAL III

## CHAPTER 09 (Vectors)

Dot product:




If $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$

& $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

$$\underline{u} \cdot \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

$$\underline{v} \cdot \underline{u} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Angle calculation

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}$$

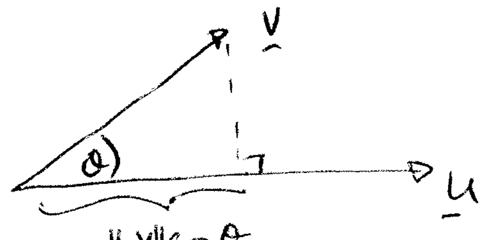
$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

$\underline{u} \times \underline{v} = -\,\underline{v} \times \underline{u}$ (negative)

$\underline{u} \cdot \underline{u} = \|\underline{u}\|^2$

$\underline{u} \times \underline{u} = \underline{0}$

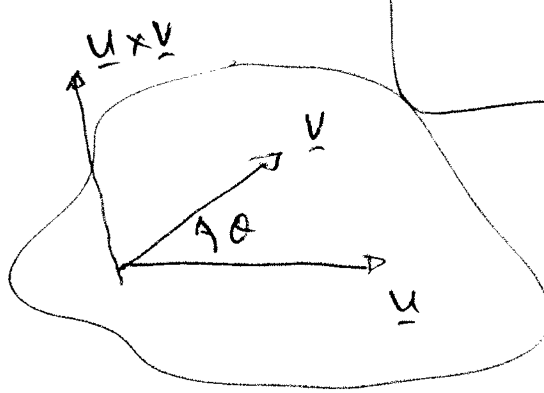
Projections




$$\text{proj}_{\underline{u}} \underline{v} = (\|\underline{v}\| \cos\theta) \frac{\underline{u}}{\|\underline{u}\|}$$

$$\Rightarrow \text{proj}_{\underline{u}} \underline{v} = \left( \frac{\underline{v} \cdot \underline{u}}{\|\underline{u}\|^2} \right) \underline{u}$$

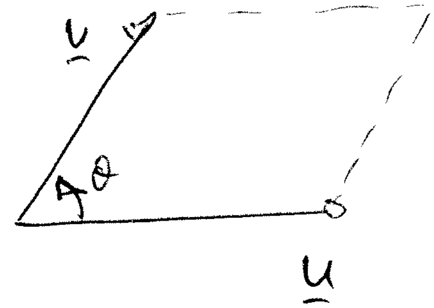
Cross product




$$\underline{u} \times \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \sin\theta \; \underline{n}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\underline{i} + (u_3 v_1 - u_1 v_3)\underline{j} + (u_1 v_2 - u_2 v_1)\underline{k}$$

## Area of a Parallelogram




$|\,\|\underline{u}\|\|\underline{v}\|\sin\theta\,|$

$= \|\underline{u} \times \underline{v}\|$

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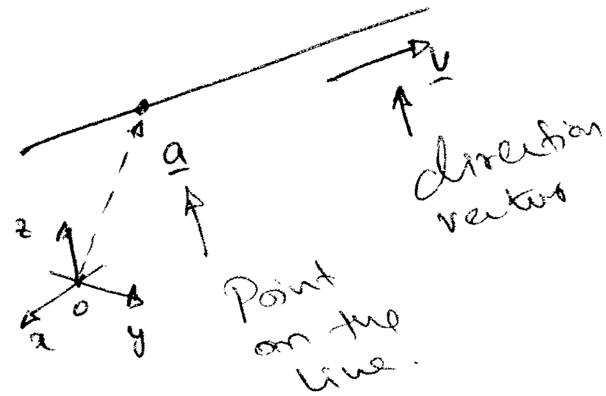
## Equation of a line




vector form: $\underline{a} + t\underline{v}$

parametric form:
$$x = a_1 + t v_1$$
$$y = a_2 + t v_2$$
$$z = a_3 + t v_3$$

Symmetric form: $\dfrac{x-a_1}{v_1} = \dfrac{y-a_2}{v_2} = \dfrac{z-a_3}{v_3}$

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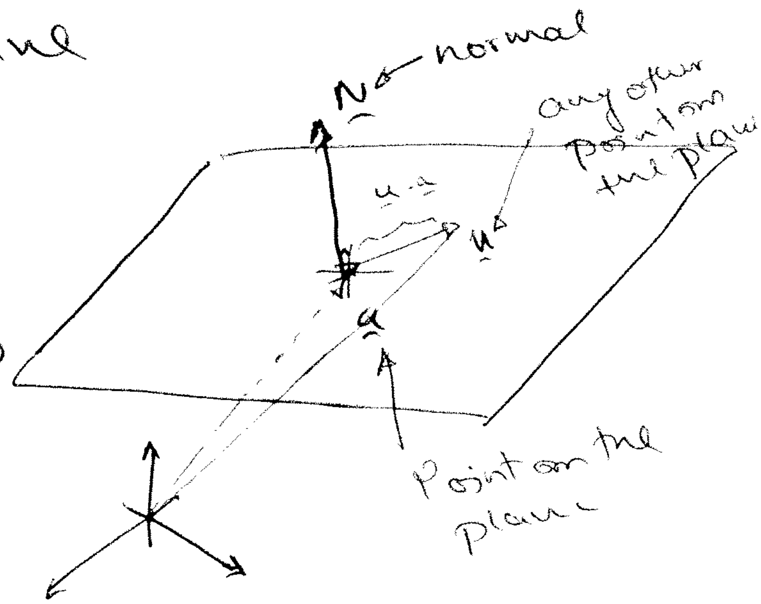
## Equation of a plane




Vector form
$$(\underline{u} - \underline{a}) \cdot \underline{N} = 0$$

Point-normal form
$$A(x-a_1) + B(y-a_2) + C(z-a_3) = 0$$

Standard form
$$Ax + By + Cz = D.$$

$$\underline{N} = A\underline{i} + B\underline{j} + C\underline{k}$$

# CHAPTER 10 (Vector functions)

## Vector valued functions:

Dot and Cross product

$$\underline{F}(t) = f_1(t)\,\underline{i} + f_2(t)\,\underline{j} + f_3(t)\,\underline{k}$$

$$\underline{G}(t) = g_1(t)\,\underline{i} + g_2(t)\,\underline{j} + g_3(t)\,\underline{k}$$

$$\underline{F}(t) \cdot \underline{G}(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)$$

$$\underline{F}(t) \times \underline{G}(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix}$$

$$= (f_2(t)g_3(t) - f_3(t)g_2(t))\,\underline{i} + (f_3(t)g_1(t) - f_1(t)g_3(t))\,\underline{j} + (f_1(t)g_2(t) - f_2(t)g_1(t)\,\underline{k}$$

---

$$\lim_{t\to t_0} \underline{F}(t) = \left(\lim_{t\to t_0} f_1(t)\right)\underline{i} + \left(\lim_{t\to t_0} f_2(t)\right)\underline{j} + \left(\lim_{t\to t_0} f_3(t)\right)\underline{k}$$

---

$$\frac{d\,F(t)}{dt} = \left(\frac{d f_1(t)}{dt}\right)\underline{i} + \left(\frac{d f_2(t)}{dt}\right)\underline{j} + \left(\frac{d f_3}{dt}\right)\underline{k}$$

---

$$\int F(t)\,dt = \left(\int f_1\,dt\right)\underline{i} + \left(\int f_2\,dt\right)\underline{j} + \left(\int f_3\,dt\right)\underline{k}$$

Applications.

$\underline{r}(t) =$ position

$\underline{v}(t) =$ Velocity

$\underline{a}(t) =$ acceleration

---

differential relations

$$\underline{a}(t) = \frac{d\underline{v}(t)}{dt} \; ; \qquad \underline{v}(t) = \frac{dr(t)}{dt}$$

$$\underline{a}(t) = \frac{d^2 r(t)}{dt}$$

---

Integral relations

$$\underline{v}(t) = \underline{v}(0) + \int_0^t \underline{a}(\tau)\, d\tau$$

$$r(t) = \underline{r}(0) + \int_0^t \underline{v}(\tau)\, d\tau$$

# Curve theory.

Arc length parameter of a curve $\underline{R}(t)$

$$s(t) = \int_{t_0}^{t} \|\underline{R}'(\tau)\| \, d\tau$$

$$\|\underline{R}'(t)\| = \sqrt{\left(\frac{dr_1}{dt}\right)^2 + \left(\frac{dr_2}{dt}\right)^2 + \left(\frac{dr_3}{dt}\right)^2}$$

$$\boxed{\underline{R}'(t) = \frac{dr_1}{dt}\underline{i} + \frac{dr_2}{dt}\underline{j} + \frac{dr_3}{dt}\underline{k}}$$

---

Using t-parameter

Unit tangent $= \underline{T}(t) = \dfrac{\underline{R}'(t)}{\|\underline{R}'(t)\|}$

Unit normal $= \underline{N}(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|}$

---

Using arc-length parameter s.

Unit tangent $= \underline{T}(s) = \dfrac{d\underline{R}(s)}{ds}$

Unit normal $= N(s) = \dfrac{1}{\kappa}\dfrac{d\underline{T}(s)}{ds}$

---

## Curvature $\kappa$.

| Using arc length | Two derivatives form (in t) | Cross derivative form |
|---|---|---|
| $\kappa(s) = \left\| \dfrac{dT(s)}{ds} \right\|$ | $\kappa(t) = \dfrac{\|T'(t)\|}{\|R'(t)\|}$ | $\kappa(t) = \dfrac{\|\underline{R}'(t) \times \underline{R}''(t)\|}{\|\underline{R}'(t)\|^3}$ |

# CHAPTER 11 (functions of several variables)

## Limits

Proving limits exists is hard!!!

To show limits does not exist:

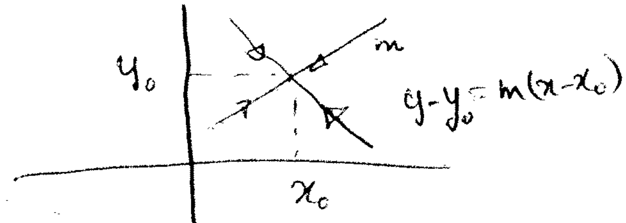
Given $f(x,y)$, to show limits does not exist: Try many paths if they do not end up with the same value, then the limit does not exist.

try the following to show that

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

(1a) $\lim_{x\to x_0} f(x, y_0)$ (1b) $\lim_{y\to y_0} f(x_0, y)$

(2) $\lim_{x\to x_0} f\left(x, m(x-x_0)+y_0\right)$




(3a) $\lim_{x\to x_0} f(x, a(x-x_0)^n+y_0)$ (3b) $\lim_{y\to y_0} f(a(y-y_0)^n+x_0, y)$

(4) "Any other" Path possible

# Partial Derivatives

Differentiate with respect to one variable assuming that the others are constant.

Eg. $f(x,y,z) = \sin^{yz} x$

$$\frac{\partial f}{\partial x} = \left(yz \sin^{(yz-1)} x\right)(\cos x).$$

---

Implicit differentiation.

Eg Given $z$ as a function of $x$ & $y$ implicitly: $x^2+y^2+z^2=0$. find $\frac{\partial^2 z}{\partial y\,\partial x}$

$\frac{\partial x}{\partial y}=0$ & $\frac{\partial y}{\partial x}=0$ here since we are assuming that they are two independant variables.

differentiating w.r.t $x$:

$$2x+0+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-x}{z}.$$

Continued...

$$\frac{\partial z}{\partial x} = \frac{-x}{z}$$ from previous page.

Similarly, partially differentiating in $y$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-x}{z}\right)$$

$$= -x\left(\frac{\partial}{\partial y}\left(\frac{1}{z}\right)\right)$$

(with respect to $y$, $x$ is constant, but $z$ is a function of $y$)

$$= -x\left(\frac{-1}{z^2}\right)\frac{\partial z}{\partial y}$$

$$= \frac{x}{z^2}\frac{\partial z}{\partial y}$$

$$= \frac{x}{z^2}\cdot\left(-\frac{y}{z}\right)$$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{z^3}}$$

Total Derivative.

given $f(x,y,z)$

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz.$$

Approximations:

$$f(x+\delta x, y+\delta x, z+\delta z) \approx f(x,y,z) + \left(\frac{\partial f}{\partial x}\right)\delta x + \left(\frac{\partial f}{\partial y}\right)\delta y + \left(\frac{\partial f}{\partial z}\right)\delta z$$

---

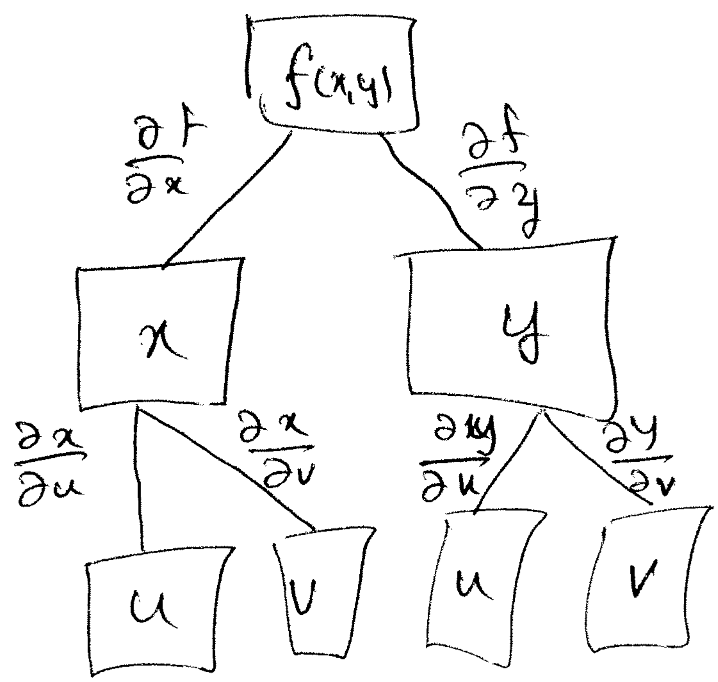
Chain rule

Eg. $f = f(x,y)$ & $x = x(u,v)$

& $y = y(u,v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$



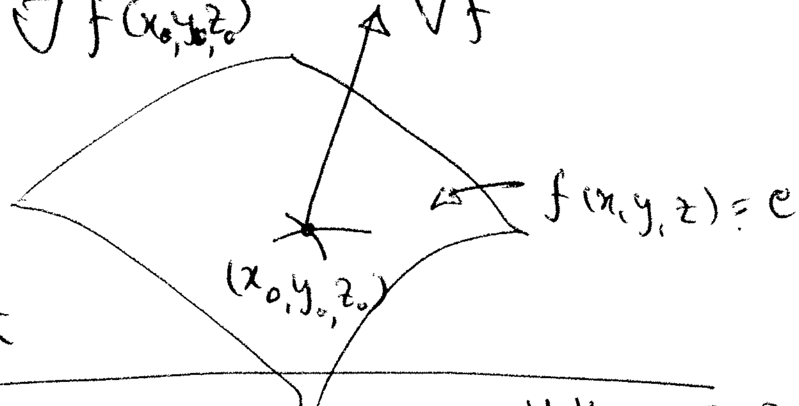
# Gradients

Given $f(x,y,z)$, ← Scalar function of $x, y$ & $z$

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

This gives a vector function!

Interpretations of $\nabla f(x_0,y_0,z_0)$




* **Normal** to the level curve $f(x,y,z)=c$ at point $x_0, y_0, z_0$

Normal line: $\dfrac{x-x_0}{f_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{f_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{f_z(x_0,y_0,z_0)}$

* Direction of the maximal increase of the function $f(x,y,z)$ at the point $(x_0,y_0,z_0)$

* $\|\nabla f\|_{(x_0,y_0,z_0)}$ gives the maximal increase of the function $f(x,y,z)$ at the point $(x_0,y_0,z_0)$

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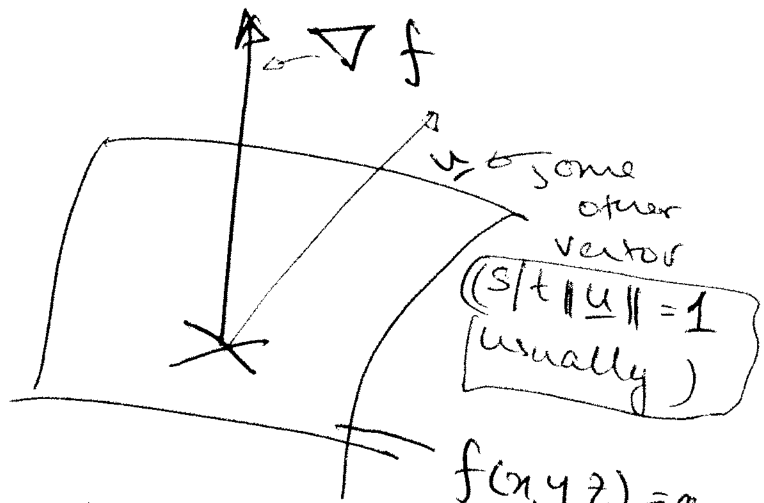
Equation of the tangent plane to $f(x,y,z)=c$ at point $(x_0,y_0,z_0)$

$$(\nabla f)\cdot\left((x-x_0)\underline{i} + (y-y_0)\underline{j} + (z-z_0)\underline{k}\right) = 0$$

$$\Rightarrow \boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0}$$

Directional derivative

$$D_u f = (\nabla f) \cdot \frac{\underline{u}}{\|\underline{u}\|}$$




This gives the variation of $f(x,y,z)$ along the direction given by $\frac{\underline{u}}{\|\underline{u}\|}$

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## Optimization

Critical point of $f(x,y)$:

* Points with **Both** $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.
* Points where $f(x,y)$ is not defined, or the partial derivatives not defined
* Boundary points.

Classification:

| $D = f_{xx} f_{yy} - f_{xy}^2$ | $f_x$ | Type |
|---|---|---|
| $+$ | $+$ | minima |
| | $-$ | maxima |
| $-$ | anything | Saddle point |
| $0$ | anything | Inconclusive |

If $D = 0$, the test fails, so, the critical point can be anything. You will have to use "some other" method to arrive at the conclusion.

# CHAPTER 12

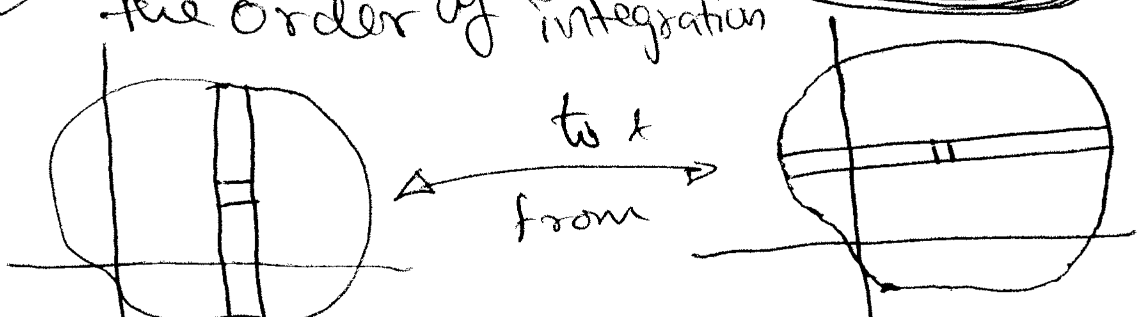
## Major Concepts:

* Setting up integrals.

Mainly "find the domain"

- Eg. • Setting up the limits
- • Set up for volume calculation
- • Set up for Surface Area calculation
  - need to find the integrand (function) as well

* Interchanging Limits ~~...~~ the order of integration




THE LIMITS WILL CHANGE

* Change of variables

mainly ~~and~~ between

* Rectangular
* Cylindrical
* Spherical
* Polar

Jacobians

MATH 2350: CALCULUS III - Spring 2011 - Quiz on Integration

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**Instructions**

• NO calculators allowed • NO formula sheets allowed • Answer on this test book only • Please write clearly
• Show all *necessary* work to earn full credit. • You may lose points if sufficient work is not shown.
• Attempt all problems • Time allowed: **10 minutes**

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Name:

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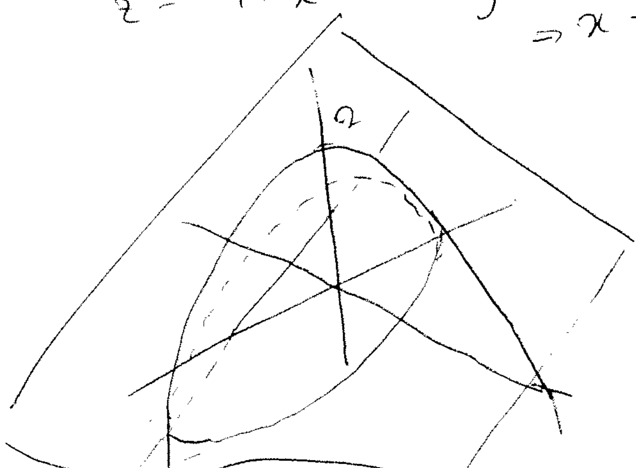
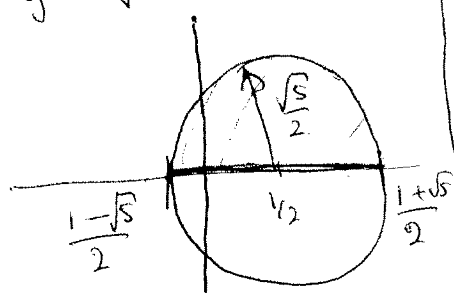
1. Set up the TRIPLE integral (do not evaluate) to find the volume of the region bound by the paraboloid $x^2+y^2+z=2$, the plane $x+z=1$ and in the region $y \geq 0$.

$z = 2 - x^2 - y^2$
$z = 1 - x$

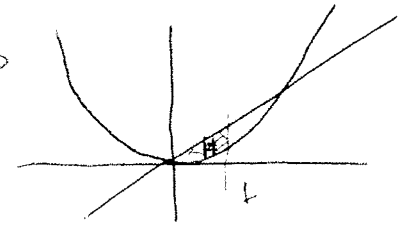
$1 - x = 2 - x^2 - y^2$
$\Rightarrow x^2 + y^2 - x = 1$
$y = \sqrt{1 - x^2 + x}$

$y^2 + (x - \frac{1}{2})^2 = 1 + \frac{1}{4}$
$y^2 + (x - \frac{1}{2})^2 = \frac{5}{4}$



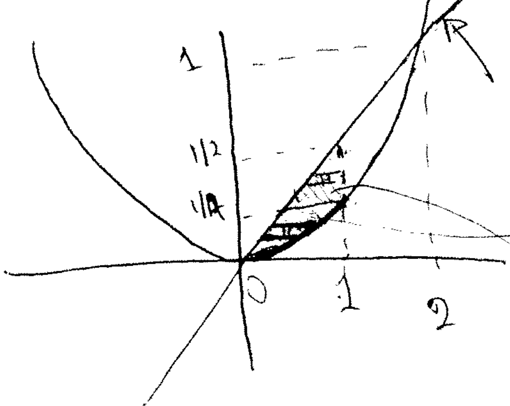



$$\boxed{\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \int_0^{\sqrt{1-x^2+x}} \int_{1-x}^{2-x^2-y^2} dz\,dy\,dx}$$



2. Given $\int_0^1 \int_{x^2/4}^{x/2} f(x,y)\,dy\,dx$, change the order of integration to $dx\,dy$.

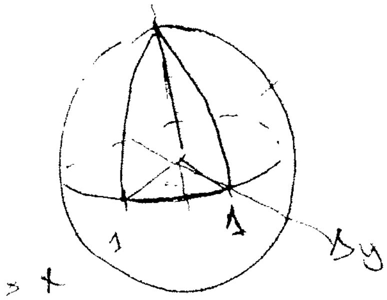
$y = \frac{x^2}{4} \Rightarrow x = 2\sqrt{y}$
$y = x/2 = x = 2y$



$$\int_0^{1/4} \int_{2y}^{2\sqrt{y}} f(x,y)\,dx\,dy + \int_{1/4}^{1/2} \int_{2y}^{1} f(x,y)\,dx\,dy$$

3. Consider the triple integral in rectangular Cartesian coordinates $\iiint\limits_D e^{x^2+y^2}\,dx\,dy\,dz$, where $D$ is the region of the unit sphere in the first octant (i.e. $x^2+y^2+z^2=1$ and $x\ge 0,\ y\ge 0,\ z\ge 0$).

Rewrite (do not evaluate) this integral in SPHERICAL coordinates.




$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} e^{\rho^2\sin\varphi}\ \underbrace{\rho^2\sin\varphi}_{\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|}\ d\rho\,d\varphi\,d\theta$$

$0\le\rho\le 1$

$0\le\theta\le\pi/2$

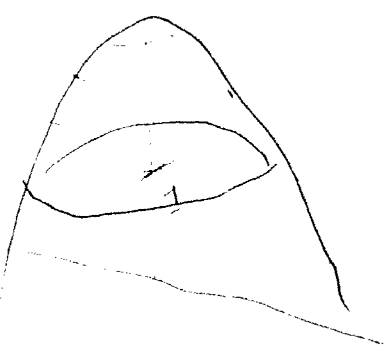
$0\le\varphi\le\frac{\pi}{2}$

for spherical

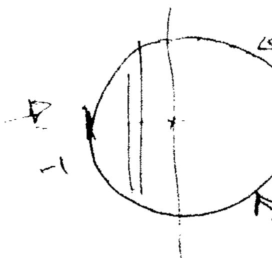
$$x^2+y^2=(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2$$
$$=\rho^2\sin^2\varphi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{=1})$$
$$=\rho^2\sin^2\varphi$$

4. *[BONUS]* Set up the integral (do not evaluate) in CYLINDRICAL coordinates to calculate the volume of the region bound by the paraboloid $x^2+y^2+z=2$ and the plane $z=1$.

$z=2-(x^2+y^2)$




$$\left.\begin{aligned}x^2+y^2+z&=2\\1&=z\end{aligned}\right\}\ \text{solve}$$




$$x^2+y^2=1$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1}^{2-x^2-y^2} dz\,dy\,dx$$

$$\int_0^{2\pi}\int_0^{1}\int_1^{2-r^2} r\ dz\,dr\,d\theta$$

$r \leftarrow \frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

$x^2+y^2=r^2$ for cylindrical

**For grading purposes only**

| 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|
| | | | | |

Surface Area formulae:

Projecting on to $xy$ plane
using $z = f(x,y)$

$$\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy$$

Projecting onto $yz$ plane
using $x = g(y,z)$

$$\iint_D \sqrt{g_y^2 + g_z^2 + 1}\, dy\,dz$$

Projecting onto $zx$ plane
using $y = h(z,x)$

$$\iint_D \sqrt{h_z^2 + h_x^2 + 1}\, dz\,dx$$

# CHAPTER 13

## Notation

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

---

For a scalar function $f = f(x, y, z)$,

$$\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

gradient of $f$ (as in chapter 11)

---

For a vector function $\underline{F}(x,y,z) = f_1(x,y,z)\underline{i} + f_2(x,y,z)\underline{j} + f_3(x,y,z)\underline{k}$

$$\nabla \cdot \underline{\mathbf{F}} = \text{div } \underline{\mathbf{F}} = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \cdot \left(f_1\underline{i} + f_2\underline{j} + f_3\underline{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla \times \underline{F} = \text{Curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\underline{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\underline{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\underline{k}$$

$$\nabla^2 = \nabla \cdot \nabla = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = \text{Laplacian}$$

Scalar function $f$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla \cdot (\nabla f)$$

$$= \text{div}(\text{grad} f).$$

---

Def. If for a scalar function $f(x,y)$ we have $\nabla^2 f = 0$, then we ~~[illegible]~~ say that $f$ is <u>Harmonic</u>.

---

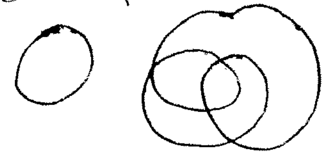
Some Useful Results. $\underline{F}$ vector, $f$ scalar

- (*) $\nabla \cdot (\nabla \times \underline{F}) = \text{div}(\text{Curl}\ \underline{F}) = 0$
- (*) $\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$
- (*) $\nabla \times (\nabla f) = 0$

# Line integrals:

$\int_C$ ← Path not closed

$\oint_C$ ← Path closed



## Scalar valued functions

$$\int_C f \, ds$$

$C$ ← Path

$ds$ ← arc length function

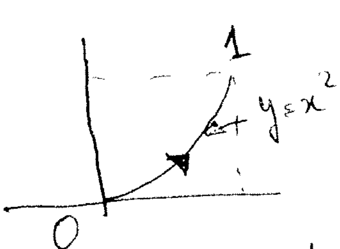
If the path can be parameterized in the form $x = x(t)$, $y = y(t)$, $z = z(t)$,

then, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

So, the integral will look like

$$\int f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Eg:




$t: 0 \to 1$

$$\boxed{\begin{aligned} x &= t \\ y &= t^2 \\ \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 2t \end{aligned}}$$

$$\int_C xy \, ds = \int_0^1 (t)(t^2)\sqrt{(1)^2 + (2t)^2}\, dt$$

$$= \int_0^1 t^3 \sqrt{4t^2 + 1}\, dt$$

## Vector valued functions

$$\int_C \underline{F} \cdot d\underline{R}$$

$\underline{R}$ ← Path parameterization

or $\int_C f_1 \, dx + f_2 \, dy + f_3 \, dz$

(where $\underline{F} = f_1 \underline{i} + f_2 \underline{j} + f_3 \underline{k}$

$\underline{R} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$)

| $\nabla \times F = \underline{0}$ | $\nabla \times F \neq 0.$ |
|---|---|
| over conservative fields (Path independant | Non-conservative field (Path dependent). |

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$$\int \underline{F} \cdot d\underline{R} = \int_C f_1\,dx + f_2\,dy + f_3\,dz$$

$\nabla \times F = \underline{0}$ (Conservative)
(Path Independant)

$$\boxed{* \quad \oint_C \underline{F} \cdot d\underline{R} = 0}$$

There is a "Scalar potential" $f$, such that $\nabla f = \underline{F}$.

Then $\int_C \underline{F} \cdot d\underline{R} = f(Q) - f(P)$

P = starting pt, Q = Ending pt.

$$\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

$$\underline{F} = f_1\underline{i} + f_2\underline{j} + f_3\underline{k}$$

Eg: $\underline{F} = (yz + \sin x)\underline{i} + (xz + e^y)\underline{j} + (xy + z^2)\underline{k}$

Then, you can show that $\nabla \times \underline{F} = 0$

Then, set $\frac{\partial f}{\partial x} = yz + \sin x$.

$$\Rightarrow f = \int yz + \sin x\,dx = xyz - \cos x + C_1(y,z)$$

$$\frac{\partial f}{\partial y} = xz + \frac{\partial C_1}{\partial y} = xz + e^y \Rightarrow \frac{\partial C_1}{\partial y} = e^y$$

$$\Rightarrow C_1 = \int e^y\,dy \Rightarrow C_1 = e^y + C_2(z)$$

$$\Rightarrow f = xyz - \cos x + e^y + C_2(z)$$

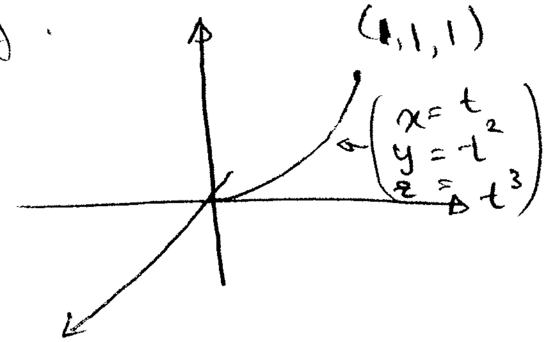
$$\frac{\partial f}{\partial z} = xy + \frac{\partial C_2}{\partial z} = xy + z^2 \Rightarrow \frac{\partial C_2}{\partial z} = z^2$$

$$\Rightarrow C_2 = \int z^2\,dz = \frac{z^3}{3} + C$$

$$\Rightarrow f = xyz - \cos x + e^y + \frac{z^3}{3} + C$$

(Non Conservative) $\nabla \times F \neq \underline{0}$
(Path dependant)

Eg.




$$\int_C xy\,dx + yz\,dy + zx\,dz$$

$$\Rightarrow \frac{dx}{dt} = 1 \Rightarrow dx = dt$$

$$\frac{dy}{dt} = 2t \Rightarrow dy = 2t\,dt$$

$$\frac{dz}{dt} = 3t^2 \Rightarrow dz = 3t^2\,dt$$

So, $\int_C xy\,dx + yz\,dy + zx\,dz$

$$= \int_0^1 (t)(t^2)\,dt + (t^2)(t^3)(2t\,dt) + (t^3)(t)(3t^2\,dt)$$

$$= \int_0^1 t^3 + 2t^5 + 3t^6\,dt$$

$\vdots$

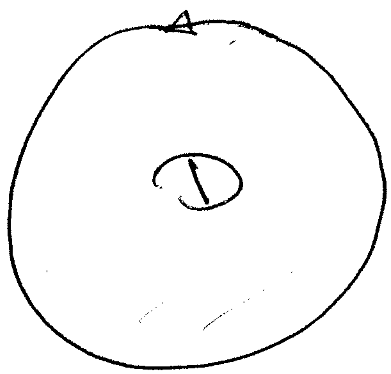
$$\boxed{\text{THIS METHOD CAN BE USED EVEN IF } \nabla \times \underline{F} = \underline{0}}$$

# GREEN'S THEOREM

for $\oint_C \underline{F} \cdot d\underline{R}$

Closed path Integrals

$$\oint_C M\,dx + N\,dy = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\,dy$$




CONDITIONS:

M & N should be differentiable in D

# Surface integrals.

## Scalar valued function

$$\iint_D g(x,y,z)\, dS$$

$g(x,y,z)$ ← Scalar function

$dS$ ← surface Element

If the surface is given by $z = f(x,y)$

$$\iint_D g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\; dx\,dy$$

## Vector valued functions

$$\iint_D \underline{F} \cdot \underline{N}\, dS$$

$\underline{F}$ ← Vector function

$\underline{N}$ ← Normal to the surface

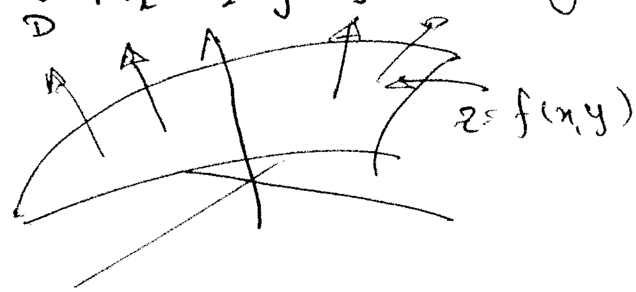
$\underline{F} = f_1\underline{i} + f_2\underline{j} + f_3\underline{k}$

If the surface is given by $z = f(x,y)$,

$$\iint_D (f_1\underline{i} + f_2\underline{j} + f_3\underline{k}) \cdot (-f_x\underline{i} - f_y\underline{j} + \underline{k})\, dx\,dy$$

$$= \iint_D -f_1 f_x - f_2 f_y + f_3 \; dx\,dy$$

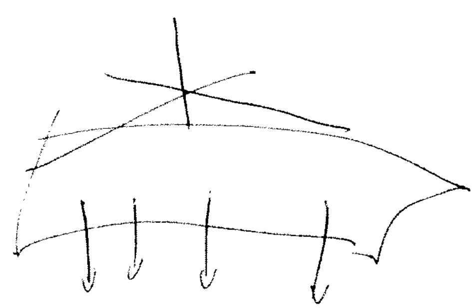
for




and

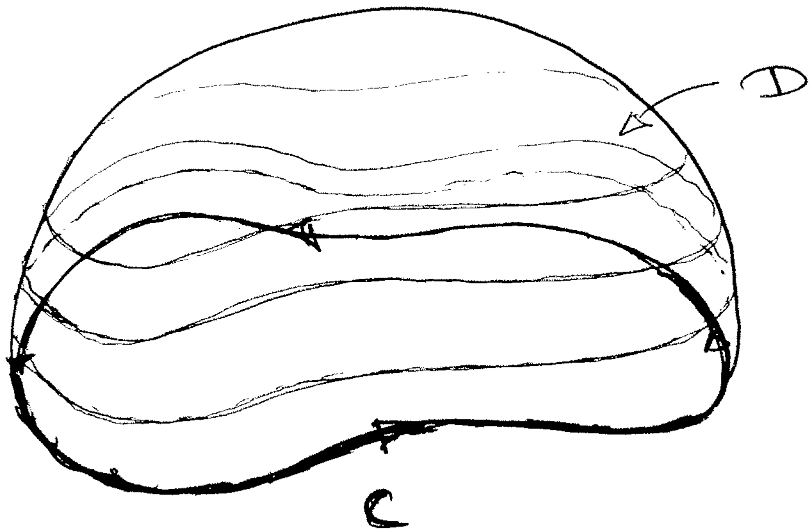
$$\iint_D f_1 f_x + f_2 f_y - f_3 \, dx\,dy$$

for



Stokes theorem.

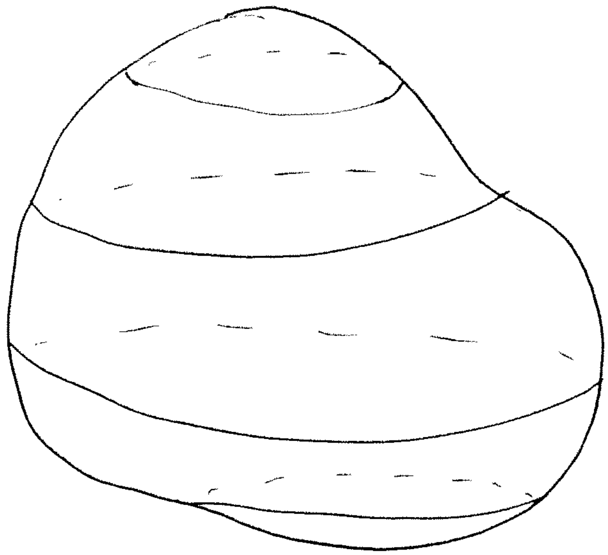
$$\oint_C \underline{F} \cdot d\underline{R} = \iint_D (\nabla \times \underline{F}) \cdot \underline{N} \, ds$$




Closed curve surface

<u>ABOVE</u> the curve.

Divergence theorem.



CLOSED SURFACE.

$$\iint_S \underline{F} \cdot \underline{N} \, dS = \iiint_D (\nabla \cdot \underline{F}) \, dV.$$