MATH 2350: CALCULUS III

Spring 2011, Sections 002 & 004

More Hints for Mid Term Test # 1 – Take Home Portion – Problem 2

Example:

Consider the dynamics of a particle on $\mathbb{R}^2$ using the time parameter t, is defined using the polar coordinates (r, θ) with $r = \frac{1}{(t+1)}$ and $\theta = \ln(t+1)$; $t \geq 0$.

1. **Write the polar equation (in the form $r = f(\theta)$) which describes trace of the path of the particle.**

   Note that from the parameterization of θ we get $t + 1 = \mathrm{e}^{\theta}$. So, $r = \dfrac{1}{\mathrm{e}^{\theta}}$. Which is simply $r = \mathrm{e}^{-\theta}$

2. **Find the parametric equation of this curve in rectangular Cartesian coordinates.**

   Using θ, it is simply $x = \mathrm{e}^{-\theta}\cos\theta$ and $y = \mathrm{e}^{-\theta}\sin\theta$.

   If we use t, it is simply $x = \frac{1}{(t+1)}\cos(\ln(t+1))$ and $y = \frac{1}{(t+1)}\sin(\ln(t+1))$.

3. **Find the tangent to this curve.**

   We can either set $\boldsymbol{R}(t) = \frac{1}{(t+1)}\cos(\ln(t+1))\boldsymbol{i} + \frac{1}{(t+1)}\sin(\ln(t+1))\boldsymbol{j}$ and directl differentiate w.r.t. t. Or else, we can use the θ parameterization and use the chain rule as follows:

$$
\begin{aligned}
\boldsymbol{R}(\theta(t)) &= \mathrm{e}^{-\theta}\cos\theta\boldsymbol{i} + \mathrm{e}^{-\theta}\sin\theta\boldsymbol{j} \\
\boldsymbol{R}'(\theta(t)) &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}\theta} \\
\boldsymbol{R}'(\theta(t)) &= \left(\frac{1}{t+1}\right)\left[\left(-\mathrm{e}^{-\theta}\cos\theta - \mathrm{e}^{-\theta}\sin\theta\right)\boldsymbol{i} + \left(-\mathrm{e}^{-\theta}\sin\theta + \mathrm{e}^{-\theta}\cos\theta\right)\boldsymbol{j}\right] \\
\boldsymbol{R}'(\theta(t)) &= \left(\frac{1}{t+1}\right)\mathrm{e}^{-\theta}\left[-\left(\cos\theta + \sin\theta\right)\boldsymbol{i} + \left(-\sin\theta + \cos\theta\right)\boldsymbol{j}\right] \\
\boldsymbol{R}'(\theta(t)) &= \left(\frac{1}{t+1}\right)^2\left[-\left(\cos(\ln(t+1)) + \sin(\ln(t+1))\right)\boldsymbol{i} + \left(-\sin(\ln(t+1)) + \cos(\ln(t+1))\right)\boldsymbol{j}\right]
\end{aligned}
$$

If you read my notes or page 655 on the book, you will see that you need to compute $r'(t)$, $r''(t)$, $\theta'(t)$ and $\theta''(t)$ for the next two sections.
So, let's calculate them beforehand: $r'(t) = \frac{-1}{(t+1)^2}$, $r''(t) = \frac{2}{(t+1)^3}$, $\theta'(t) = \frac{1}{(t+1)}$ and $\theta''(t) = \frac{-1}{(t+1)^2}$.

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4. **Find the velocity components of this particle along $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ directions**

   From my notes or from the book you can see that:
   Velocity component along $\boldsymbol{u}_r$ is $r'(t) = \frac{-1}{(t+1)^2}$

   Velocity component along $\boldsymbol{u}_\theta$ is $r(t)\theta'(t) = \left(\frac{1}{(t+1)}\right)\left(\frac{1}{(t+1)}\right) = \frac{1}{(t+1)^2}$

5. **Find the acceleration components of this particle along $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ directions**

   Again from my notes or from the book you can see that:
   Acceleration component along $\boldsymbol{u}_r$ is $r''(t) - r\,(\theta'(t))^2 = \frac{-1}{(t+1)^3} - \left(\frac{1}{(t+1)}\right)\left(\frac{1}{(t+1)}\right)^2 = \frac{-2}{(t+1)^3}$

   Acceleration component along $\boldsymbol{u}_\theta$ is $r(t)\theta''(t) + 2r'\theta'(t) = \frac{1}{(t+1)}\frac{-1}{(t+1)^2} - 2\frac{-1}{(t+1)^2}\frac{1}{(t+1)} = \frac{-3}{(t+1)^3}$

6. **At a given time t, calculate how far away the particle is from the origin.**

   Clearly The distance from the origin is $\|\boldsymbol{R}(t) - (0\boldsymbol{i} + 0\boldsymbol{j})\| = \|\boldsymbol{R}(t)\| = r(t) = \frac{1}{(t+1)}$

7. **Find what happens to the acceleration, velocity and distance from the origin as $t \to \infty$.**

   It is very easy to see that the distance from the origin goes to zero as $t \to \infty$.

   Note that the components of velocity and acceleration along $\boldsymbol{u}_r$ as well as along $\boldsymbol{u}_\theta$ goes to zero as $t \to \infty$.
   So, both the acceleration as well as the velocity goes to zero $t \to \infty$.

There could be some typos but the general idea is correct