HW #3

Complete solutions for odd numbered problems

§10.1

① $\underline{F}(t) = 2t\,\underline{i} - 3t\,\underline{j} + \frac{1}{t}\underline{k}$

Defined for all t

not defined (becomes ∞) for t = 0.

Domain : All reals except t = 0.

(This can be written as $t \neq 0$ or more correctly as $\mathbb{R}\setminus\{0\}$.)

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③ $\underline{F}(t) = \sin t\,\underline{i} + \cos t\,\underline{j} + \tan t\,\underline{k}$

Defined for all t

not defined for odd multiples of $\pi/2$

Domain All reals except $t = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

(This can be written as $t \neq (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

or better, $\mathbb{R}\setminus\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$.

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⑤ $h(t)\underline{f}(t) = \frac{\sin t}{\cos t}\underline{i} + \frac{\sin t}{\sin t}\underline{j} + \frac{\sin t}{\tan t}\underline{k}$

$= \tan t\,\underline{i} + 1\,\underline{j} + \cos t\,\underline{k}$

Domain : $t \neq (2n+1)\frac{\pi}{2}$ IS IT ? NO

CORRECT SOLUTION GIVEN ON NEXT PAGE.

$\boxed{\text{Pay attention to the box in page 636}}$

Domain of $h(t) = \sin t \Rightarrow$ all real numbers

Domain of $\underline{F}(t) = \left(\frac{1}{\cos t}\right)^2 + \left(\frac{1}{\sin t}\right)^2 + \left(\frac{1}{\tan t}\right)^2$

Not defined for integer multiples of π i.e. even multiples of $\frac{\pi}{2}$

not defined for odd multiples of $\frac{\pi}{2}$.

$\therefore$ Domain of $\underline{F}(t)$ is all real numbers exept <u>all</u> integer multiples of $\frac{\pi}{2}$. i.e. $\mathbb{R} \setminus \{\frac{n\pi}{2} ; n \in \mathbb{Z}\}$.
or $t \neq \frac{n\pi}{2} ; n \in \mathbb{Z}$.

$\therefore$ The intersection of domain of $h(t)$ and the domain of $\underline{F}(t)$ is $t \neq \frac{n\pi}{2} ; n \in \mathbb{Z}$

(7) Domain of $\underline{F}(t) = \ln t\, \underline{i} + 3t - t^2 \underline{k}$

is $t > 0$.

Domain of $\underline{G}(t) = 1\underline{i} + 5t\underline{j} - t^2\underline{k}$

is all real numbers.

$\therefore$ The Domain of $\underline{F}(t) - \underline{G}(t)$ is the intersection of the two domains:

i.e. $t > 0$.

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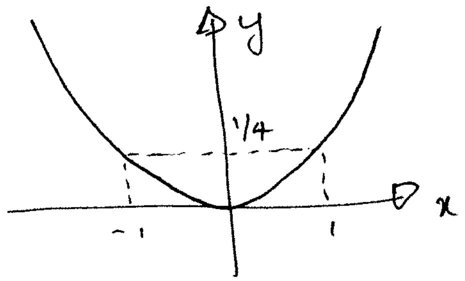
(9) $\underline{f}(t) = 2t\,\underline{i} + t^2\,\underline{j}$

So, $\underbrace{x = 2t}_{t = \frac{x}{2}}$, $y = t^2$.

$t = \frac{x}{2} \longrightarrow y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$



we need to check the range of values x and y can take:

x can take any real number and y can take only positive real numbers.

Not complete yet,

Extra:

Suppose, you were given that,

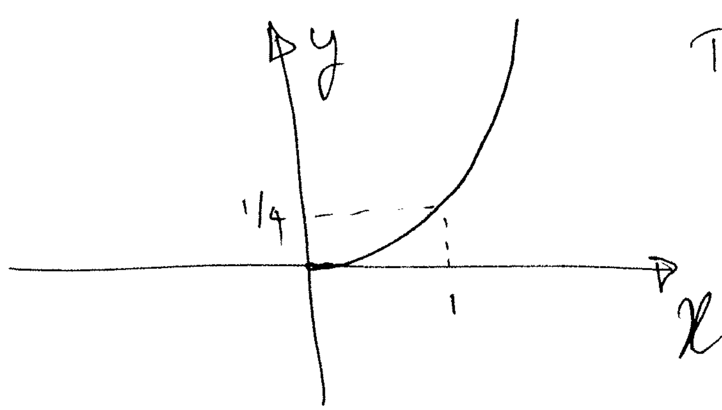
$\underline{f}(t) = 2t^2 \underline{i} + t^4 \underline{j}$.

then, $\underbrace{x = 2t^2}$, $y = t^4$.

$t^2 = \frac{x}{2} \longrightarrow y = \left(\frac{x}{2}\right)^2$

so again $y = \frac{x^2}{4}$

But this time, we see that x can only take positive real numbers so, the graph will look like:



The positive x side of the graph of $y = \frac{x^2}{4}$

Another one:

Suppose $\underline{f}(t) = 2\sqrt{(1-t)(1+t)}\,\underline{i} + (1-t)(1+t)\,\underline{j}$

"Clearly" the domain of $\underline{f}(t)$ is $-1 \le t \le 1$

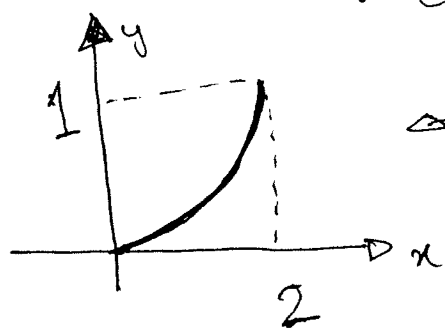
and we can "easily" show that

this gives the equation $y = \frac{x^2}{4}$ (again).

but, x-variable can take only the

value $0 \le x \le 2$ (why?)

$t = \pm 1$ $\quad$ $t = 0$



← This is the whole graph!

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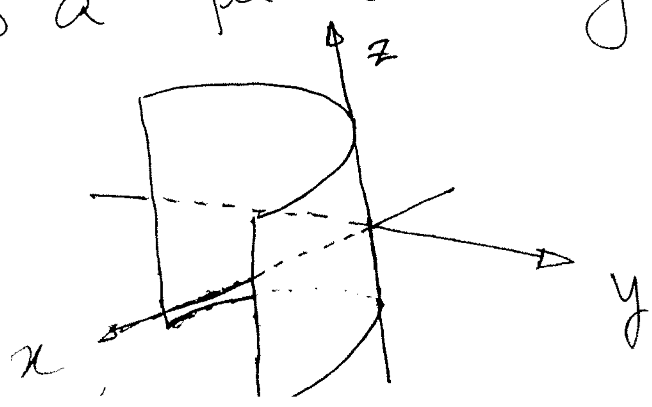
9 Cont'd...

Since the $\underline{k}$ vector can change freely,

This gives a parabolic cylinder

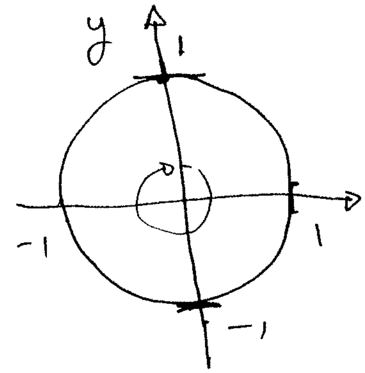


(11) $\underline{G}(t) = \underbrace{\sin t}_{x}\, \underline{i} - \underbrace{\cos t}_{y}\, \underline{j}$

$x = \sin t$; $y = -\cos t$,

$x^2 + y^2 = \sin^2 t + (-\cos t)^2 = \sin^2 t + \cos^2 t = 1$.

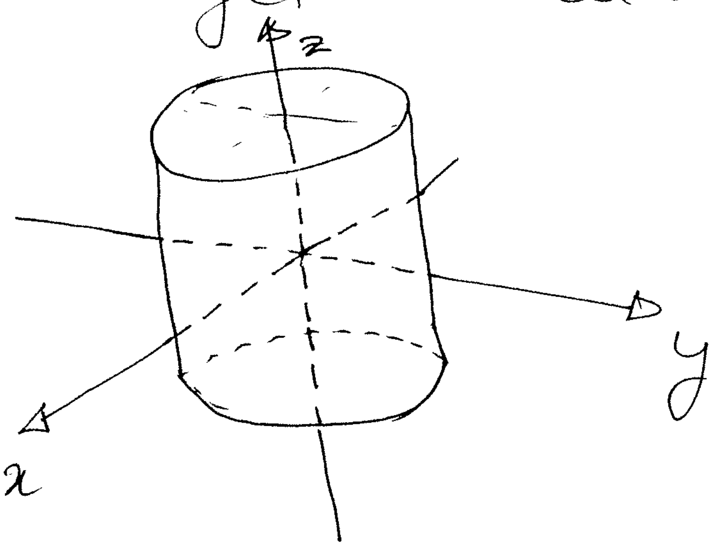
So, $x^2 + y^2 = 1 \Rightarrow$ Circle



at $t = 0$, $\underline{G}(0) = 0\underline{i} - 1\underline{j}$

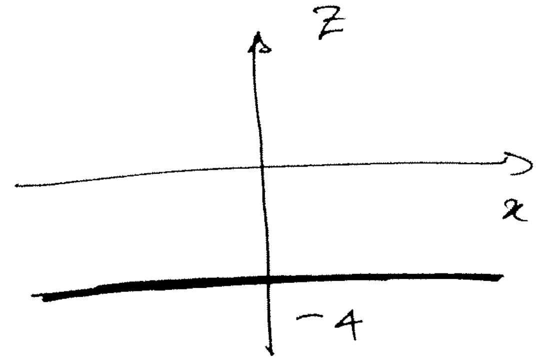
$\underline{G}\left(\frac{\pi}{2}\right) = 1\underline{i} - 0\underline{j}$

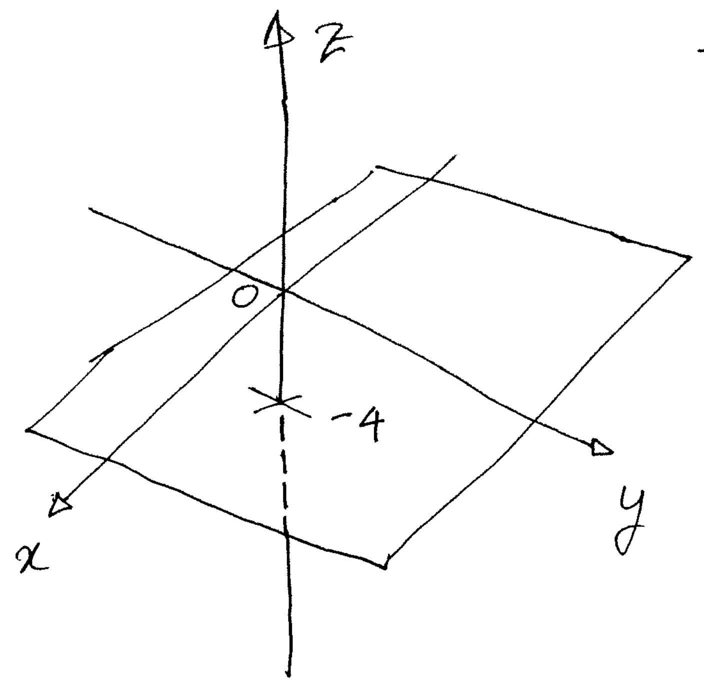
Again, the $\underline{k}$ vector is free.
So, we get a circular cylinder



(13) $\underline{r}(t) = t\underline{i} - 4\underline{k}$

So, $x = t$; $z = -4$





A plane parallel to the x-y plane and passes through $z = -4$

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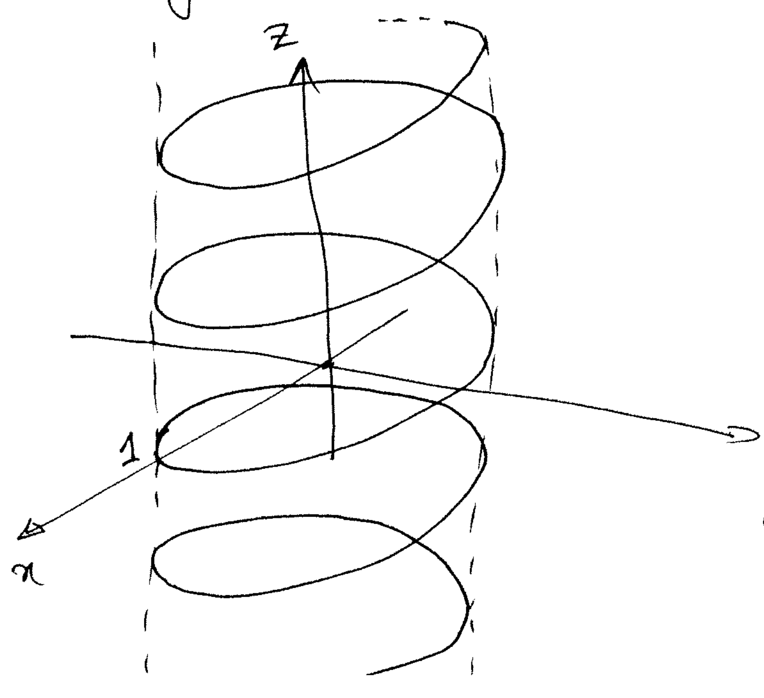
(15) $\underline{r}(t) = \cos t\,\underline{i} + \sin t\,\underline{j} + t\underline{k}$

$x = \cos t$; $y = \sin t$, $z = t$.

$x^2 + y^2 = \underline{1}$

Helix



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$y = x^2$; $z = 2$.

~~Sol~~ let $x = t \Rightarrow y = t^2$.

$\therefore$ vector valued function:

$\underline{f}(t) = t\,\underline{i} + t^2\,\underline{j} + 2\,\underline{k}$

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39 $x = 2t$; $y = 1 - t$; $z = \sin t$.

$\underline{f}(t) = 2t\,\underline{i} + (1-t)\,\underline{j} + \sin t\,\underline{k}$

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41 $z = \sqrt{9 - x^2 - y^2}$; $x = y^2$

let $y = t$; then, $x = t^2$,

$z = \sqrt{9 - (t^2)^2 - t^2} = \sqrt{9 - t^2 - t^4}$

So, $\underline{f}(t) = t^2\,\underline{i} + t\,\underline{j} + \sqrt{9 - t^2 - t^4}\,\underline{k}$

(43) (a) ~~Se~~ $x = \sin t$; $y = \cos t$; $z = t$,

~~K2H.~~ Need $z = x^2 + y^2$.

L.H.S. $= z = t$.

R.H.S. $= x^2 + y^2 = \sin^2 t + \cos^2 t = 1$

So, L.H.S. $\neq$ R.H.S.

Hence, we can conclude that, the curve given by $\underline{f}(t) = \sin t\, \underline{i} + \cos t\, \underline{j} + t\underline{k}$ does not lie on the surface $z = x^2 + y^2$

(b) ~~&~~ (c) are left for you to try.

(44) Intersection of $x^2+3y^2=1$; & $z = 2x^2-1$.

$\boxed{\text{Key point: } \cos^2\theta + \sin^2 t = 1}$

So, let $x = \cos t$, ~~and~~,

$y = \frac{1}{\sqrt{3}} \sin t$

Then, $x^2 + 3y^2 = \cos^2 t + \cancel{3} \frac{\sin^2 t}{\cancel{3}} = 1$ ✓

So, $z = 2\cos^2 t - 1$

$z = \cos 2t$

Recall that
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

So, the curve of intersection.

$\underline{r}(t) = \cos t\, \underline{i} + \frac{1}{\sqrt{3}} \sin t\, \underline{j} + \cos 2t\, \underline{k}$

$\boxed{\text{Note: This is an ad-hoc method!}}$

(45) $\lim_{t \to 1} [2t\underline{i} - 3\underline{j} + e^t \underline{k}]$

$= 2\underline{i} - 3\underline{j} + e^1 \underline{k}$ (since all functions are continuous)

(47)

$$\lim_{t \to 0} \left[\frac{\sin t}{t^2+t-1} \underline{i} - \frac{t}{t^2+t-1} k \right]$$

$$= \left(\lim_{t \to 0} \underbrace{\frac{\sin t}{t^2+t-1}}_{0/0 \text{ form}} \right) \underline{i} + \underbrace{\left(\lim_{t \to 0} \frac{-t}{t^2+t-1} \right)}_{= \infty} \underline{k}$$

use L'hôpital's rule

$$\lim_{t \to 0} \frac{\cos t}{2t+1} = \infty$$

Book says the answer is $\underline{0}$

Clearly there is an obvious typo

$$\lim_{t \to \infty} \left[\frac{\sin t}{t^2+t-1} \underline{i} - \frac{t}{t^2+t-1} \underline{k} \right]$$

Then $\left(\lim_{t \to \infty} \frac{\sin t}{t^2+t-1} \right) \underline{i} + \left(\lim_{t \to \infty} \frac{-t}{t^2+t-1} \right) \underline{k}$

The $\sin t$ function is bounded $-1 \le \sin t \le 1$
but $t^2+t-1 \longrightarrow \infty$ as $t \to \infty$
$\therefore \lim_{t \to \infty} \frac{\sin t}{t^2+t-1} = 0$. so, we get $\underline{0}$

(49) $\lim_{t\to 0}\left[\frac{te^t}{1-e^t}\,\underline{i}+\frac{e^{(t-1)}}{\cos t}\,\underline{j}\right]$

$$\lim_{t\to 0}\underbrace{\frac{te^t}{1-e^t}}_{0/0 \text{ form}} = \lim_{t\to 0}\frac{te^t+e^t}{-e^t} = \lim_{t\to 0} -(t+1) = -\underline{1}$$

L'Hôpital

and $\lim_{t\to 0}\frac{e^{t-1}}{\cos t} = e^{-1}$

So, $\lim_{t\to 0}\left[\frac{te^t}{1-e^t}\,\underline{i}+\frac{e^{t-1}}{\cos t}\,\underline{j}\right]$

$= -1\underline{i} + e^{-1}\underline{j}$

(S1) $\lim_{t\to 0^+}\left[\frac{\sin 3t}{\sin 2t}\,\underline{i} + \frac{\ln(\sin t)}{\ln(\tan t)}\,\underline{j} + t\ln t\,\underline{k}\right]$

$= \left[\lim_{t\to 0^+}\frac{\sin 3t}{\sin 2t}\right]\underline{i} + \left[\lim_{t\to 0^+}\frac{\ln(\sin t)}{\ln(\tan t)}\right]\underline{j} + (t\ln t)\,\underline{k}$

$= \frac{3}{2}\,\underline{i} + 1\,\underline{j} + 0\,\underline{k}$

Note that

$$\lim_{t\to 0^+}\frac{\sin 3t}{\sin 2t} = \lim_{t\to 0^+}\frac{3\cos 3t}{2\cos 2t} = \frac{3}{2}$$

0/0 form, L' Hôpital

$$\lim_{t\to 0^+}\frac{\ln(\sin t)}{\ln(\tan t)} = \lim_{t\to 0^+}\frac{\left(\frac{\cos t}{\sin t}\right)}{\left(\frac{\sec^2 t}{\tan t}\right)}$$

$\frac{\infty}{\infty}$ form, L' Hôpital

$$= \lim_{t\to 0^+}\left(\frac{\cos t}{\sin t}\cdot\frac{\tan t}{\sec^2 t}\right)$$

$$= \lim_{t\to 0^+}\left(\frac{\cos t}{\cancel{\sin t}}\cdot\frac{\cancel{\sin t}}{\cos t}\cdot\cos^2 t\right)$$

$$= \underline{1}$$

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{(1/t)} = \lim_{t\to 0^+}\frac{1/t}{-1/t^2} = \lim_{t\to 0^+}\frac{-1}{t}\cdot\frac{t^2}{1} = 0$$

$\frac{\infty}{\infty}$ form, L' Hôpital

(53) $\underline{f}(t) = t\underline{i} + 3\underline{j} - (1-t)\underline{k}$

Continuous for all $t \in \mathbb{R}$.

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(55) $$\underline{G}(t) = \frac{1\underline{i} + 2\underline{j}}{(t^2+t)} = \frac{1\underline{i} + 2\underline{j}}{t\,(t+1)}$$

$$= \frac{1}{t(t+1)}\,\underline{i} + \frac{2}{t(t+1)}\,\underline{j}$$

The denominator becomes zero for $t=0$ or $t=-1$.

So Domain is when, $t \neq 0$ and $t \neq -1$

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(57) $$\underline{F}(t) = te^t\,\underline{i} + \frac{e^t}{t}\,\underline{j} + 3e^t\,\underline{k}$$

Continuous for all t

Not continuous at $t=0$.

$\therefore$ $\underline{F}(t)$ is continuous for $t \neq 0$.

(59) $\underline{R}(t) = t\underline{i} + \left(\frac{1-t}{t}\right)\underline{j} + \left(\frac{1-t^2}{t}\right)\underline{k}$

We can write the equation of a plane if we know 3 points which do not lie on a line.

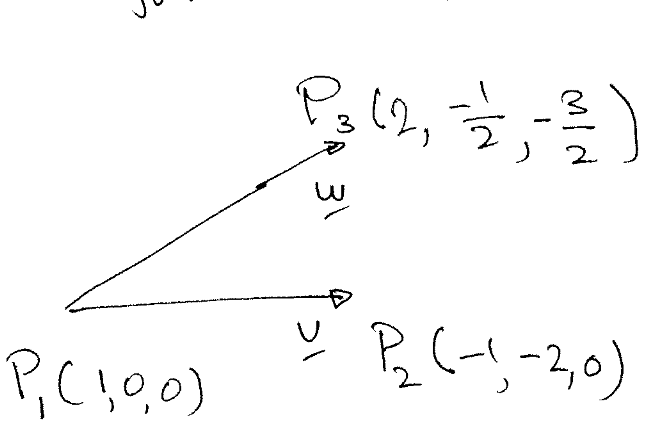
Since we are told that the curve is on a plane we can start by finding 3 points:

Note that the domain of $\underline{R}(t)$ is $t \neq 0$.

For $t = 1 \Rightarrow \underline{R}(1) = 1\underline{i} + 0\underline{j} + 0\underline{k} \Rightarrow P_1$

for $t = -1 \Rightarrow \underline{R}(-1) = -1\underline{i} - 2\underline{j} + 0\underline{k} \Rightarrow P_2$

for $t = 2 \Rightarrow \underline{R}(2) = 2\underline{i} - \frac{1}{2}\underline{j} - \frac{3}{2}\underline{k} \Rightarrow P_3$



$\underline{v} = -2\underline{i} - 2\underline{j} + 0\underline{k}$

$\underline{w} = 1\underline{i} - \frac{1}{2}\underline{j} - \frac{3}{2}\underline{k}$

Clearly $\underline{v}$ and $\underline{w}$ are not along the same direction.

$$\underline{N} = \underline{V} \times \underline{W} = \begin{vmatrix} i & j & k \\ -2 & -2 & 0 \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{vmatrix}$$

$$= 3\underline{i} - 3\underline{j} + 3\underline{k}$$

So, plane:

$$3(x-1) - 3y + 3z = 0$$

So, $3x - 3y + 3z - 3 = 0$

or Simply, $x - y + z - 1 = 0$

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61 - Show the work!

§ 10.2

(1) $\underline{F}(t) = t\underline{i} + t^2\underline{j} + (t+t^3)\underline{k}$

$\underline{F}'(t) = 1\underline{i} + 2t\underline{j} + (1+3t^2)\underline{k}$

(3) $\underline{F}(s) = s\ln s\,\underline{i} + s\ln s\,\underline{j} - e^s \ln s\,\underline{k}$

$\underline{F}'(s) = \left(\ln s + \frac{s}{s}\right)\underline{i} + \frac{s}{s}\underline{j} - \left(e^s\ln s + \frac{e^s}{s}\right)\underline{k}$

$\underline{F}'(s) = (\ln s + 1)\underline{i} + \frac{s}{s}\underline{j} - \left(e^s \ln s + \frac{e^s}{s}\right)\underline{k}$

(5), (7), similar

(9) $f(x) = \underbrace{\left[x\underline{i} + (x+1)\underline{j}\right]}_{\underline{F}_1(x)} \cdot \underbrace{\left[2x\underline{i} - 3x^2\underline{j}\right]}_{\underline{F}_2(x)}$

$f'(x) = \underline{F}_1'(x) \cdot \underline{F}_2(x) + \underline{F}_1(x) \cdot \underline{F}_2'(x)$

$= (1\underline{i} + 1\underline{j}) \cdot (2x\underline{i} - 3x^2\underline{j})$

$\quad + (x\underline{i} + (x+1)\underline{j}) \cdot (2\underline{i} - 6x\underline{j})$

$= 2x - 3x^2 + 2x - 6x^2 - 6x = -2x - 9x^2$

or $f'(x) = \frac{d}{dx}\left[2x^2 - 3x^3 - 3x^2\right] = \frac{d}{dx}\left[-x^2 - 3x^3\right]$

$= -2x - 9x^2$

(11) $g(x) = \| \sin x\, \underline{i} - 2x\, \underline{j} + \cos x\, \underline{k} \|$

$g(x) = \sqrt{\sin^2 x + 4x^2 + \cos^2 x}$

$g(x) = \sqrt{1+4x^2} = (1+4x^2)^{1/2}$

$g'(x) = \frac{1}{2} \cdot \frac{8x}{\sqrt{(1+4x^2)}} = \frac{4x}{\sqrt{1+4x^2}}$

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(13) $\underline{R}(t) = t\underline{i} + t^2\underline{j} + 2t\underline{k}$; $t = 1$.

velocity $= \underline{v}(t) = \underline{R}'(t) = 1\underline{i} + 2t\underline{j} + 2\underline{k}$

acceleration $= \underline{a}(t) = \underline{v}'(t) = \underline{R}''(t) = 0\underline{i} + 2\underline{j} + 0\underline{k}$

Speed $= \|\underline{v}(t)\| = \|\underline{R}'(t)\| = \sqrt{1 + 4t^2 + 4} = \sqrt{5+4t^2}$

direction of motion $\underline{d}(t) = \frac{\underline{v}(t)}{\|\underline{v}(t)\|} = \frac{1\underline{i} + 2t\underline{j} + 2\underline{k}}{\sqrt{5+4t^2}}$

So, $\underline{v}(1) = 1\underline{i} + 2\underline{j} + 2\underline{k}$

$\underline{a}(1) = 0\underline{i} + 2\underline{j} + 0\underline{k}$

$\|\underline{v}(1)\| = \sqrt{5+4} = 3$

$\underline{d}(1) = \frac{1}{3}\underline{i} + \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$

15, 17 similar

(19) $\underline{F}(t) = t^2 \underline{i} + 2t \underline{j} + (t^3 + t^2)\underline{k}$

Tangent to $\underline{F}(t) = \underline{F}'(t) = 2t\underline{i} + 2\underline{j} + (3t^2 + 2t)\underline{k}$

$\underline{F}'(0) = 0\underline{i} + 2\underline{j} + 0\underline{k}$

$\underline{F}'(1) = 2\underline{i} + 2\underline{j} + 5\underline{k}$

$\underline{F}'(-1) = -2\underline{i} + 2\underline{j} + 1\underline{k}$

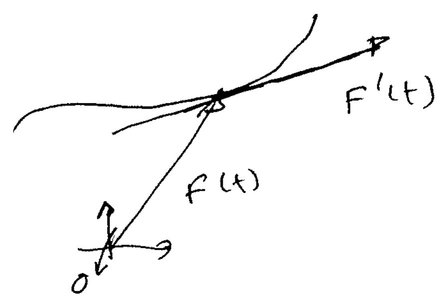
(21) similar.

(23) $\underline{F}(t) = t^{-3}\underline{i} + t^{-2}\underline{j} + t^{-1}\underline{k}$

$\underline{F}(-1) = -1\underline{i} + 1\underline{j} - 1\underline{k}$

$\underline{F}'(t) = -3t^{-4}\underline{i} - 2t^{-3}\underline{j} - t^{-2}\underline{k}$

$\underline{F}'(-1) = -3\underline{i} + 2\underline{j} - 1\underline{k}$



Equation of the line:

$$\frac{x+1}{-3} = \frac{y-1}{2} = \frac{z+1}{-1}$$

parametric equation: (using parameter r, let's say)

$(-1-3r)\underline{i} + (1+2r)\underline{j} + (-1-r)\underline{k}$

(25) $\int t\,\underline{i} - e^{3t}\,\underline{j} + 3\,\underline{k}\;dt$.

$$= \frac{t^2}{2}\underline{i} - \frac{e^{3t}}{3}\underline{j} + 3t\,\underline{k} + \underline{c} \leftarrow \text{vector}$$

(27), ~~similar~~

$$\int \ln t\,\underline{i} - t\,\underline{j} + 3\,\underline{j} = (t\ln|t| - t)\underline{i} - \frac{t^2}{2}\underline{j} + 3t\,\underline{k} + \underline{c} \leftarrow \text{vector.}$$

(from tables). Appendix D (A 21)

(29)

$$\int t\ln t\,\underline{i} - \sin(1-t)\,\underline{j} + t\,\underline{k}\;dt$$

$$= \frac{t^2}{2}\left(\ln|t| - \frac{1}{2}\right)\underline{i} - \cos(1-t)\,\underline{j} + \frac{t^2}{2} + \underline{c} \quad (A\ 31)$$

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(31) $\underline{R}(t) - \underline{R}(0) = \int_{\tau=0}^{t} \underline{U}(\tau)\,d\tau = \int_0^t \tau^2\,\underline{i} - e^{2\tau}\,\underline{j} + \sqrt{\tau}\,k\,d\tau$

$\underline{R}(0) = 1\underline{i} + 4\underline{j} - 1\underline{k}$

$$= \frac{\tau^3}{3}\Big|_0^t\,\underline{i} - \frac{e^{2\tau}}{2}\Big|_0^t + \frac{2}{3}\tau^{3/2}\Big|_0^t$$

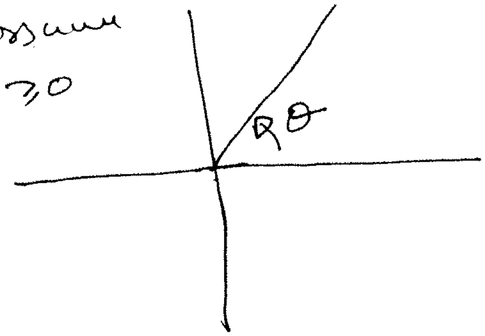
$$= \frac{t^3}{3}\,\underline{i} + \left(\frac{1-e^{2t}}{2}\right)\underline{j} + \frac{2}{3}(t^{3/2})\,\underline{k}$$

So, $R(t) = \left(\frac{t^3}{3} + 1\right)\underline{i} - \frac{1}{2}(9 + e^{2t})\,\underline{j} + \left(\frac{2}{3}t^{3/2} - 1\right)\underline{k}$

(33) Similar

§ 10.3

(9) $x = 2t \; ; \; y = t$. assume $t \geq 0$



So, $y = \dfrac{x}{2}$

$\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$. Constant.

$r = \sqrt{x^2+y^2} = \sqrt{4t^2+t^2} = \sqrt{5}\, t$.

So, $r' = \sqrt{5}, \quad r'' = 0,$

$\theta' = 0, \quad \theta'' = 0$

Therefore, $\underline{v}(t) = \sqrt{5}\,\underline{u}_r + 0\,\underline{u}_\theta$

$\underline{a}(t) = 0\,\underline{u}_r + 0\,\underline{u}_\theta$

$$\boxed{\begin{aligned} \underline{v}(t) &= r'\,\underline{u}_r + r\theta'\,\underline{u}_\theta \\ \underline{a}(t) &= \left(r'' - r(\theta')^2\right)\underline{u}_r + \left(r\theta'' + 2r'\theta'\right)\underline{u}_\theta \end{aligned}}$$

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(10) $x = \sin t, \quad y = \cos t$

$r = \sqrt{x^2+y^2} = 1$

$\theta = t$

$\left.\begin{aligned} r' &= 0 \\ r'' &= 0 \\ \theta' &= 1 \\ \theta'' &= 0 \end{aligned}\right.$

So, $\underline{v}(t) = 0\,\underline{u}_r + 1\,\underline{u}_\theta \; ; \; \underline{a}(t) = -1\,\underline{u}_r + 0\,\underline{u}_\theta$

(11) $r = \sin\theta, \quad \theta = 2t.$

i.e. $r = \sin(2t) \quad ; \theta = 2t.$

$r'(t) = 2\cos 2t \quad ; \quad r''(t) = -4\sin 2t,$

$\theta'(t) = 2 \quad ; \quad \theta''(t) = 0.$

$\underline{V}(t) = 2\cos 2t \ \underline{u}_r + 2\sin 2t \ u_\theta$

$\underline{a}(t) = (-4\sin 2t - 4\sin 2t)\underline{u}_r$

$\quad + (\sin 2t \cdot 0 + (2)(2\cos 2t)(2))\underline{u}_\theta$

$= -8\sin 2t \ \underline{u}_r + 8\cos 2t \ \underline{u}_\theta$

---

(13) $r = 5(1+\cos\theta) \ ; \ \theta = 2t+1$

So, $r = 5 + 5\cos(2t+1)$

$r'(t) = -10\sin(2t+1); \ r''(t) = -20\cos(2t+1)$

$\theta'(t) = 2 \quad ; \theta'' = 0$

$\underline{V}(t) = -10\sin(2t+1)\ \underline{u}_r + 10(1+\cos(2t+1))\underline{u}_\theta$

$\underline{a}(t) = [-20\cos(2t+1) - 20(1+\cos(2t+1))]\underline{u}_r$

$\quad + (0 + (2)(-10\sin(2t+1))(2))u_\theta$

$= [-40\cos(2t+1) - 20]\underline{u}_r - 40\sin(2t+1)\ u_\theta$

§ 10.5

(1) $\underline{R}(t) = t^2\underline{i} + t^3\underline{j} \quad t \neq 0$

$\underline{R}'(t) = 2t\underline{i} + 3t^2\underline{j} \Rightarrow \|\underline{R}'(t)\| = \sqrt{4t^2+9t^4} = |t|\sqrt{4+9t^2}$

$\underline{T}(t) = \dfrac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \dfrac{2t}{|t|\sqrt{4+9t^2}}\underline{i} + \dfrac{3t^2}{|t|\sqrt{4+9t^2}}$

for $t > 0$,

$\underline{T}(t) = \dfrac{2}{\sqrt{4+9t^2}}\underline{i} + \dfrac{3t}{\sqrt{4+9t^2}}\underline{j}$ ← The book only gives this one

for $t < 0$

$\underline{T}(t) = \dfrac{-2}{\sqrt{4+9t^2}}\underline{i} + \dfrac{3t}{\sqrt{4+9t^2}}\underline{j}$

for $t>0$

$\underline{T}'(t) = \dfrac{2 \cdot \cancel{18}^{9} t}{-\cancel{2}(4+9t^2)^{3/2}}\underline{i} + \dfrac{\sqrt{4+9t^2}\cdot 3 - \frac{3t}{2}\cdot\frac{18t}{\sqrt{4+9t^2}}}{(4+9t^2)}\underline{j}$

$= \dfrac{-18t}{(4+9t^2)^{3/2}}\underline{i} + \dfrac{6(4+9t^2) - 54t^2}{2(4+9t^2)^{3/2}}$

$\underline{T}'(t) = \dfrac{-18t}{(4+9t^2)^{3/2}} + \dfrac{12}{(4+9t^2)^{3/2}}$

$18 = 6 \times 3$
$12 = 6 \times 2$

$\|\underline{T}'(t)\| = \dfrac{6}{(4+9t^2)^{3/2}}\left(\sqrt{9t^2+4}\right)$

$= \dfrac{6}{4+9t^2}$

Check:
$\underline{T}(t)\cdot\underline{N}(t) = 0$

$\therefore \underline{N}(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|} = \dfrac{-3t}{\sqrt{4+9t^2}}\underline{i} + \dfrac{2}{\sqrt{4+9t^2}}\underline{j}$

for $t<0$, you will get $\dfrac{3t}{\sqrt{4+9t^2}}\underline{i} + \dfrac{2}{\sqrt{4+9t^2}}\underline{j}$

(3) $\underline{R}(t) = e^t \cos t\, \underline{i} + e^t \sin t\, \underline{j} \to \underline{R}'(t) = e^t(\cos t - \sin t)\underline{i} + e^t(\sin t + \cos t)\underline{j}$

$$\|\underline{R}'(t)\| = e^t \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{=1} - 2\cos t \cancel{\sin t} + \underbrace{\cos^2 t + \sin^2 t}_{=1} + 2\cancel{\cos t} \sin t}$$

$$\|\underline{R}'(t)\| = \sqrt{2}\, e^t$$

So, $\underline{T}(t) = \dfrac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \dfrac{(\cos t - \sin t)}{\sqrt{2}}\underline{i} + \dfrac{(\sin t + \cos t)}{\sqrt{2}}\underline{j}$

$$\underline{T}'(t) = \frac{1}{\sqrt{2}}(-\sin t - \cos t)\underline{i} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\underline{j}$$

$$\|\underline{T}'(t)\| = \frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$$

So, $\underline{N}(t) = \dfrac{-1}{\sqrt{2}}(\sin t + \cos t)\underline{i} + \dfrac{1}{\sqrt{2}}(\cos t - \sin t)$.

---

(5) We have done this.

---

(7) $\underline{R}(t) = \ln t\, \underline{i} + t^2 \underline{k}$; $t > 0$.

$$\underline{R}'(t) = \frac{1}{t}\underline{i} + 2t\underline{k} \to \|\underline{R}'(t)\| = \sqrt{\frac{1}{t^2} + 4t^2} = \frac{1}{t}\sqrt{1+4t^4}$$

$$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \frac{1}{\sqrt{1+4t^4}}\underline{i} + \frac{2t^2}{\sqrt{1+4t^4}}\underline{j}$$

$$\underline{T}'(t) = \frac{-\cancel{16}^{8} t^3}{\cancel{2}(1+4t^4)^{3/2}}\underline{i} + \frac{4t\sqrt{1+4t^4} - 2t^2 \cdot \frac{4t^3}{\cancel{2}\sqrt{1+4t^4}}}{\left(\sqrt{1+4t^4}\right)^2}\underline{j} = \frac{16t^3}{(1+4t^4)^{3/2}}\underline{i} + \frac{4t + \cancel{4t^5} - \cancel{4t^5}}{(1+4t^4)^{3/2}}\underline{j}$$

$$= \frac{-8t^3}{(1+4t^4)^{3/2}}\underline{i} + \frac{4t}{(1+4t^4)^{3/2}}\underline{j}$$

$$\|\underline{T}'(t)\| = \frac{4t}{(1+4t^4)^{3/2}}\left(4t^4+1\right) = \frac{4t}{(1+4t^4)}$$

So, $\underline{N}(t) = \dfrac{-2t^2}{\sqrt{(1+4t^4)}}\underline{i} + \dfrac{1}{\sqrt{(1+4t^4)}}\underline{j}$

(9) $\underline{R}(t) = 2t\,\underline{i} + t\,\underline{j}$; over $[0, 4]$

$\underline{R}'(t) = 2\underline{i} + 1\underline{j}$

$\|\underline{R}'(t)\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\text{arc length} = \int_0^4 \|\underline{R}'(t)\|\, dt = \int_0^4 \sqrt{5}\, dt$$
$$= 4\sqrt{5}$$

---

(11) $\underline{R}(t) = 3t\,\underline{i} + 3\cos t\,\underline{j} + 3\sin t\,\underline{k}$ over $[0, \frac{\pi}{2}]$

$\underline{R}'(t) = 3\underline{i} - 3\sin t\,\underline{j} + 3\cos t\,\underline{k}$

$\|\underline{R}'(t)\| = \sqrt{3^2 + (-3\sin t)^2 + (3\cos t)^2}$

$= \sqrt{9 + 9\underbrace{(\sin^2 t + \cos^2 t)}_{=1}}$

$\|\underline{R}'(t)\| = 3\sqrt{2}$

$$\text{arc length} = \int_0^{\pi/2} 3\sqrt{2}\, dt = 3\sqrt{2} \cdot \frac{\pi}{2} = \frac{3\pi}{\sqrt{2}}$$

---

(13) $\underline{R}(t) = 4\cos t\,\underline{i} + 4\sin t\,\underline{j} + 5t\,\underline{k}$ over $[0, \pi]$

$\underline{R}'(t) = -4\sin t\,\underline{i} + 4\cos t\,\underline{j} + 5\underline{k}$

$\|\underline{R}'(t)\| = \sqrt{16\underbrace{(\sin^2 t + \cos^2 t)}_{=1} + 25} = \sqrt{41}$

So, $\text{arc length} = \int_0^{\pi} \sqrt{41}\, dt = \pi\sqrt{41}$.

(15) $y = 4x - 2$ at $x = 2$.

You can try this in ~~two~~ many ways:

Method I: using table 10.2,

$$K = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}} \quad ; \; f(x) = 4x-2.$$

$$f'(x) = 4, \quad f'' = 0$$

So $K = 0$.

Method II: Note that this is a line.

So, does it curve? No, So, the Curvature is ... $k = 0$.

Method III: Use the Parametric Equation

let $x = t$, then, $y = 4t - 2$,

i.e. $\underline{R}(t) = t\underline{i} + (4t-2)\underline{j}$, with this equation, (if you look at table 10.2, we have 4 options: Arc length parameter, Two derivatives, Cross derivatives form, or the parametric form

The easiest will be to use the two derivatives form: first find $\underline{R}'(t)$ and $\underline{T}'(t)$:

$\underline{R}'(t) = 1\,\underline{i} + 4\underline{j} \rightarrow \|\underline{R}'(t)\| = \sqrt{1+4^2} = \sqrt{17}$

$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \frac{1}{\sqrt{17}}\underline{i} + \frac{4}{\sqrt{17}}\underline{j} \qquad \underline{T}'(t) = 0\underline{i} + 0\underline{j}$

So $k = \|\underline{T}'(t)\| / \|\underline{R}'(t)\| = 0$.

17. $y = x + \frac{1}{x}$ at $x = 1$.

Functional form: $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 + \left(\frac{-1}{x^2}\right) = 1 - \frac{1}{x^2} \Rightarrow f'(1) = 0$$

$$f''(x) = \frac{2}{x^3} \Rightarrow f''(1) = 2.$$

So, $$K = \left.\frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}\right|_{x=1} = \frac{2}{(1+0^2)^{3/2}} = 2.$$

ALT. Let $x = t$ then, $y = t + \frac{1}{t}$. ; $x = 1 \Rightarrow t = 1$

So, $\underline{R}(t) = t\,\underline{i} + (t + \frac{1}{t})\,\underline{j}$

$$\underline{R}'(t) = 1\,\underline{i} + \left(1 - \frac{1}{t^2}\right)\underline{j} \Rightarrow \underline{R}'(1) = 1\,\underline{i} + 0\,\underline{j} \Rightarrow \|\underline{R}'(1)\| = 0$$

$$\|\underline{R}'(t)\| = \sqrt{1 + (1 - \frac{1}{t^2})^2} = \sqrt{1 + 1 + \frac{1}{t^4} - \frac{2}{t^2}}$$
$$= \sqrt{2 + \frac{1}{t^4} - \frac{2}{t^2}} = \frac{1}{t^2}\sqrt{2t^4 + 1 - 2t^2}$$

$$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \frac{t^2}{\sqrt{2t^4+1-2t^2}}\,\underline{i} + \frac{(t^2-1)/\cancel{t^2}}{\frac{1}{\cancel{t^2}}\sqrt{2t^4+1-2t^2}}\,\underline{j}$$

$$= \frac{t^2}{\sqrt{2t^4+1-2t^2}}\,\underline{i} + \frac{(t^2-1)}{\sqrt{2t^4+1-2t^2}}\,\underline{j}$$

First try $$\frac{d}{dt}\left[\frac{t^2}{\sqrt{2t^4+1-2t^2}}\right] = \frac{2t\left(\sqrt{2t^4+1-2t^2}\right) - t^2\cdot\left(\frac{8t^3-4t}{2\sqrt{2t^4+1-2t^2}}\right)}{(2t^4+1-2t^2)}$$

Evaluate at $t = 1 \Rightarrow \frac{2 - (4-2)}{1} = 0$

Then, $$\frac{d}{dt}\left[\frac{1}{\sqrt{2t^4+1-2t^2}}\right] = -\frac{1}{2}\,\frac{(8t^3-4t)}{(2t^4+1-2t^2)^{3/2}} \xrightarrow[\text{at } t=1]{\text{Evaluated}} -2.$$

$$\underline{T}'(1) = 0\,\underline{i} + 2\,\underline{j} \Rightarrow \|\underline{T}'(1)\| = 2$$

$$K(1) = \|T'(1)\| / \|R'(1)\| = 2/1 = 2.$$ (Same as above)

(19) $y = \ln x$; $x = 1$. $\to$ at $x=1$, $y=0$

$y = f(x) = \ln x$. $\to$ $f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} \Rightarrow \kappa(1) = \frac{|-1|}{(1+1^2)^{3/2}} = \frac{1}{2^{3/2}} = 2^{-3/2}$$

ALT. ~~Let~~ you could take $x = t$, or better take $y = t$, then, $x = e^y = e^t$.

at $x=1$ (or $y=0$), $t=0 \to x=1$ clearly)

So, $\underline{R}(t) = e^t \underline{i} + t\underline{j}$

$\underline{R}'(t) = e^t \underline{i} + 1\,\underline{j}$; $\underline{R}'(0) = 1\underline{i} + 1\underline{j}$

$\|\underline{R}'(t)\| = \sqrt{1+e^{2t}} \Rightarrow \|\underline{R}'(0)\| = \sqrt{2}$

$$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \frac{e^t}{\sqrt{1+e^{2t}}}\underline{i} + \frac{1}{\sqrt{e^{2t}+1}}\underline{j}$$

$$\underline{T}(t) = \frac{1}{\sqrt{e^{-2t}+1}}\underline{i} + \frac{1}{\sqrt{e^{2t}+1}}\underline{j}$$

$$\underline{T}'(t) = -\frac{1}{2}\cdot\frac{e^{-2t}\cdot(-2)}{[\sqrt{(e^{-2t}+1)}]^3}\underline{i} - \frac{1}{2}\cdot\frac{e^{2t}(2)}{[\sqrt{(e^{2t}+1)}]^3}\underline{j}$$

$$\underline{T}'(0) = \frac{1}{2^{3/2}}\underline{i} - \frac{1}{2^{3/2}}\underline{j}$$

$$\|\underline{T}'(0)\| = \frac{1}{2^{3/2}}\cdot\sqrt{2} = 2^{-3/2}\sqrt{2}$$

So, $$\kappa(0) = \frac{\|\underline{T}'(0)\|}{\|\underline{R}'(0)\|} = \frac{2^{-3/2}\sqrt{2}}{\sqrt{2}} = 2^{-3/2}$$

(21) $\underline{R}(t) = e^{-t}\underline{i} + 1\underline{j} + e^{-t}\underline{k} \quad ; t \geq 0$

$\underline{R}'(t) = -e^{-t}\underline{i} + 0\underline{j} + e^{-t}\underline{k}$

$\|\underline{R}'(t)\| = \sqrt{e^{-2t} + e^{-2t}} = e^{-t}\sqrt{2}$.

$s(t) = \int_0^t \sqrt{2}e^{-\tau}\,d\tau = \left[-\sqrt{2}e^{-\tau}\right]_0^t = \sqrt{2}(1-e^{-t})$

So, $s = \sqrt{2}(1-e^{-t})$

$\therefore \quad \frac{s}{\sqrt{2}} = 1 - e^{-t} \Rightarrow e^{-t} = 1 - \frac{s}{\sqrt{2}} = \frac{\sqrt{2}-s}{\sqrt{2}}$

Hence, $\tilde{\underline{R}}(s) = \frac{\sqrt{2}-s}{\sqrt{2}}\underline{i} + 1\underline{j} - \frac{\sqrt{2}-s}{\sqrt{2}}\underline{k}$

---

(23) $\underline{R}(t) = (2-3t)\underline{i} + (1+t)\underline{j} - (4t)\underline{k}$

$R'(t) = -3\underline{i} + 1\underline{j} - 4\underline{k}$

$\|\underline{R}'(t)\| = \sqrt{9+1+16} = \sqrt{26}$

$s(t) = \int_0^t \sqrt{26}\,d\tau = \sqrt{26}\,t. \Rightarrow s = \sqrt{26}\,t$

$t = \frac{s}{\sqrt{26}}$

So, $\tilde{\underline{R}}(s) = \left(2 - \frac{3s}{\sqrt{26}}\right)\underline{i} + \left(1 + \frac{s}{\sqrt{26}}\right)\underline{j} - \frac{4s}{\sqrt{26}}\underline{k}$

(25) $\underline{R}(t) = \underline{u} + t\,\underline{v}$

$\underline{R}'(t) = \underline{v}$ ← Costant vector

$\|\underline{R}'(t)\| = \|\underline{v}\|$ ← Scalar constant.

$$s(t) = \int_0^t \|\underline{v}\|\,d\tau = \|\underline{v}\|\,t.$$

So, $s = \|v\|\,t \implies t = \dfrac{s}{\|v\|}$

$$\underline{\tilde{R}}(s) = \underline{u} + s.\frac{\underline{v}}{\|\underline{v}\|}$$

So, $\underline{\tilde{T}}(s) = \underline{\tilde{R}}'(s) = \dfrac{\underline{v}}{\|\underline{v}\|}$ ← vector, independent of s.

$$\frac{d\tilde{T}}{ds} = \underline{\tilde{T}}'(s) = \underline{0}$$

$$\kappa = \left\|\frac{d\tilde{T}}{ds}\right\| = \|\underline{0}\| = 0$$

(27) $\underline{R}(t) = \ln(\sin t)\,\underline{i} + \ln(\cos t)\,\underline{j}$

$$\underline{R}'(t) = \frac{\cos t}{\sin t}\,\underline{i} + \frac{-\sin t}{\cos t}\,\underline{j}$$

$$\underline{R}'(t) = \cot t\,\underline{i} - \tan t\,\underline{j}$$

$$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|}$$

So, at $t = \pi/3$, $\tan t = \sqrt{3}$, $\cot t = \frac{1}{\sqrt{3}}$

$$\underline{R}'\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\,\underline{i} - \sqrt{3}\,\underline{j} = \frac{1}{\sqrt{3}}(1\underline{i} - 3\underline{j})$$

$$\therefore \left\|\underline{R}'\left(\frac{\pi}{3}\right)\right\| = \frac{1}{\sqrt{3}}(1+3^2)^{1/2} = \frac{\sqrt{10}}{\sqrt{3}}.$$

$$\therefore \underline{T}\left(\frac{\pi}{3}\right) = \frac{\frac{1}{\sqrt{3}}(1\underline{i} - 3\underline{j})}{\frac{\sqrt{10}}{\sqrt{3}}}$$

$$\underline{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{10}}(1\underline{i} - 3\underline{j})$$

(29) $\underline{R}(t) = \sin t\, \underline{i} + \cos t\, \underline{j} + t\underline{k}$

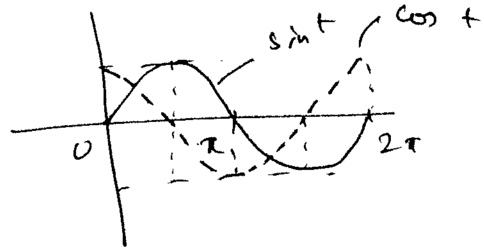
$\underline{R}'(t) = \cos t\, \underline{i} - \sin t\, \underline{j} + 1\, \underline{k}$

$\|\underline{R}'(t)\| = \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{=1} + 1} = \sqrt{2}.$

$\underline{T}(t) = \dfrac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \dfrac{1}{\sqrt{2}} \cos t\, \underline{i} - \dfrac{1}{\sqrt{2}} \sin t\, \underline{j} + \underline{1}\, \underline{k}$

(a)

$\underline{T}(\pi) = -\dfrac{1}{\sqrt{2}} \underline{i} + 0\underline{j} + 1\, \underline{k}$



$\underline{T}'(t) = -\dfrac{1}{\sqrt{2}} \sin t\, \underline{i} - \dfrac{1}{\sqrt{2}} \cos t\, \underline{j} + 0\, \underline{k}$

(b) $\|\underline{T}'(t)\| = \dfrac{1}{\sqrt{2}}$

so, $\kappa = \dfrac{\|T'(t)\|}{\|R'(t)\|} = \dfrac{\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}} = \dfrac{1}{2}.$

$S = \displaystyle\int_{\tau=0}^{\tau=\pi} \|\underline{R}'(\tau)\|\, d\tau = \int_0^{\pi} \sqrt{2}\, d\tau = \sqrt{2}\,\pi$

(31) Ellipse $9x^2 + 4y^2 = 36$.

This requires the knowledge of the parametric equation of an ellipse:

first note that, by dividing out by 36,

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \implies \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

so, let $\frac{x}{2} = \cos t$ & $\frac{y}{3} = \sin t$,

(then we get $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$).

So, the parametric equation is,

$$\underline{R}(t) = 2\cos t\, \underline{i} + 3\sin t\, \underline{j}$$

$$\underline{R}'(t) = -2\sin t\, \underline{i} + 3\cos t\, \underline{j}$$

$$\|\underline{R}'(t)\| = \sqrt{4\sin^2 t + \underbrace{9\cos^2 t}_{4\cos^2 t + 5\cos^2 t}} = \sqrt{4 + 5\cos^2 t}$$

$$\underline{T}(t) = \frac{\underline{R}'(t)}{\|\underline{R}'(t)\|} = \frac{-2\sin t}{\sqrt{4+5\cos^2 t}}\, \underline{i} + \frac{3\cos t}{\sqrt{4+5\cos^2 t}}\, \underline{j}$$

$$T'(t) = \frac{-2\sqrt{4+5\cos^2 t}\,\cos t - 2\sin t\left(\frac{-10\cos t\sin t}{2\sqrt{4+5\cos^2 t}}\right)}{\left(\sqrt{4+5\cos^2 t}\right)^2}\,\underline{i}$$

$\vdots$

(similar ugly and long term) $\underline{j}$

$\vdots$

$$\underline{K}(t) = \frac{\|\underline{T}'(t)\|}{\|\underline{R}'(t)\|},$$

differentiate $K(t)$ w.r.t t.

$\frac{dK}{dt} = 0$, find maximum & minimum.

take the 1st derivative test or the second derivative test

find t,

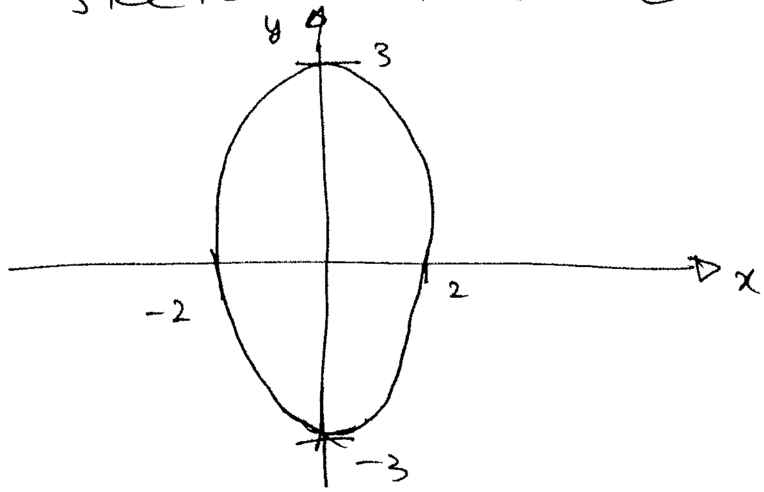
$\vdots$

TOO HARD!

No need to do all that!

See next page

First sketch the ellipse!



What is Measured by "Curvature"?

i.e. What does Curvature mean?

By the interpretation of the term, it is clear that the maximum curvature is ~~when~~ at the points $(0,3)$ and $(0,-3)$!

[32 is much easier to calculate, well, I have given the outline...]

Hint for 32. Use the functional form or the two derivatives form.

(33) $y = x^6 - 3x^2 = f(x)$

$$\frac{dy}{dx} = 6x^5 - 6x = f'(x).$$

set to 0.

$$6x^5 - 6x = 0 \Rightarrow 6x(x^4 - 1) = 0.$$

$\Rightarrow$ ~~Extrema~~ $\Rightarrow 6x(x^2-1)(x^2+1) = 0$

$\Rightarrow 6x(x-1)(x+1)(x^2+1) = 0$

Extrema: $x = -1, 0, 1.$

$$\frac{d^2y}{dx^2} = 30x^4 - 6. = f''(x)$$

$$k = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

Radius of curvature

$\boxed{\rho = \frac{1}{k}}$.

$x = 0 \Rightarrow k(0) = \frac{|-6|}{[1 + 0^2]^{3/2}} = 6 \Rightarrow \rho(0) = \frac{1}{6}$

$x = 1 \Rightarrow k(1) = \frac{|30-6|}{[1 + 0^2]^{3/2}} = 24$

$\Rightarrow \rho(1) = \frac{1}{24}$

$x = -1 \Rightarrow k(-1) = \frac{|30-6|}{(1 + 0^2)^{3/2}} = 24$

$\Rightarrow \rho(-1) = \frac{1}{24}$