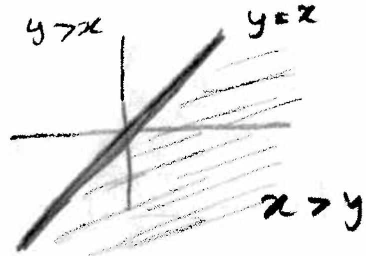
11.1

(5) $f(x,y) = \sqrt{x-y}$

Domain: $\{(x,y): x-y \geq 0\}$

That is same as $\{(x,y): x \geq y\}$



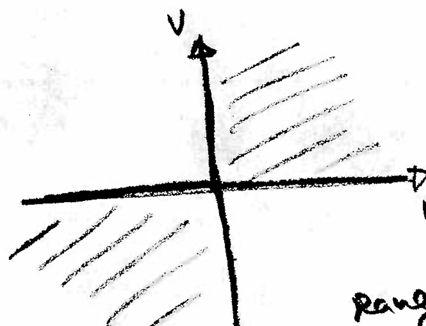
Range: $f(x,y) \geq 0$

---

(7) $f(u,v) = \sqrt{uv}$

Domain: $\{(u,v): uv \geq 0\}$

i.e u & v should be of the same sign

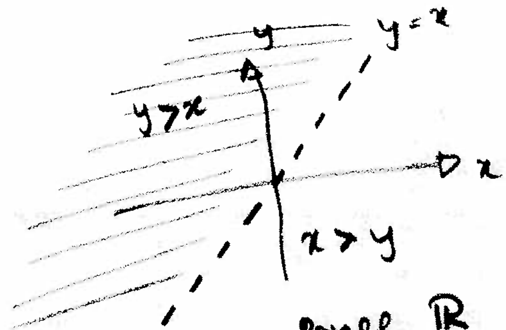


Range $f(u,v) \geq 0$

---

(9) $f(x,y) = \ln(y-x)$

Domain $\{(x,y): y-x > 0\}$



Range $\mathbb{R}$

---

(11) $f(x,y) = \sqrt{(x+3)^2 + (y-1)^2}$

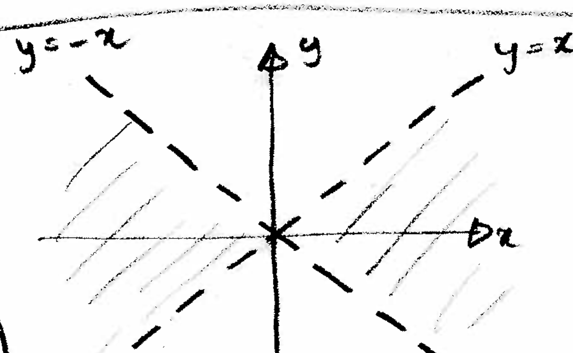
Note that for any x & y this is defined, so, domain is $\mathbb{R}^2$

Range: $f(x,y) \geq 0$

---

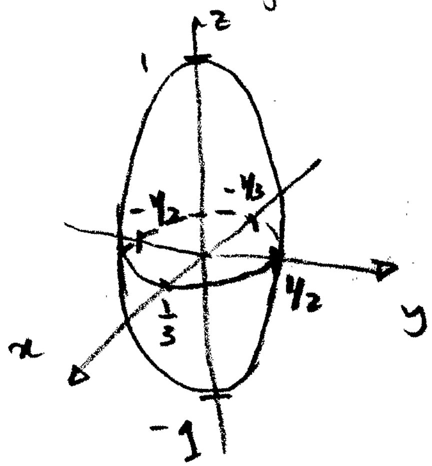
(13) $f(x,y) = \dfrac{1}{\sqrt{x^2-y^2}}$

Domain: $\{(x,y): x^2-y^2 > 0\}$



Range $f(x,y) > 0$

(27) $9x^2+4y^2+z^2=1$



Set $z=c_1$

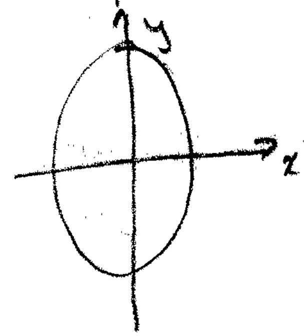
$9x^2+4y^2+c_1^2=1$

$9x^2+4y^2=1-c_1^2$

If $|c_1|<1$

$$\frac{x^2}{(1-c_1^2)/3^2}+\frac{y^2}{(1-c_1^2)/2^2}=1$$

Ellipse



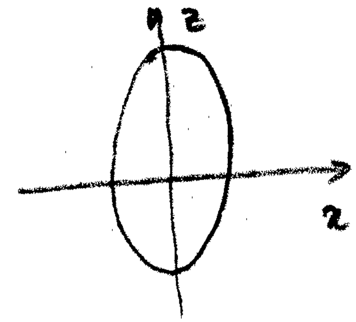
Set $y=c_2$

$9x^2+4c_2^2+z^2=1$

$9x^2+z^2=1-4c_2^2$

If $|c_2|<\frac{1}{2}$

$$\frac{x^2}{(1-4c_2^2)/3^2}+\frac{z^2}{(1-4c_2^2)}=1$$



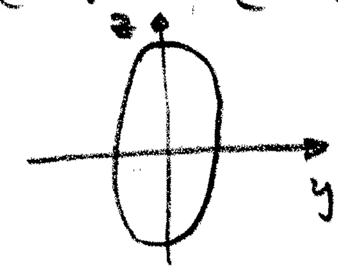
Set $x=c_3$

$9c_3^2+4y^2+z^2=1$

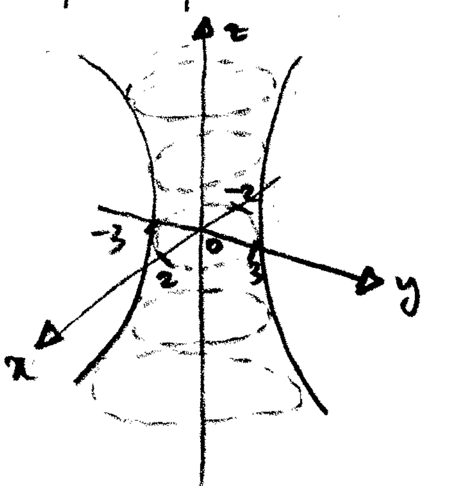
$4y^2+z^2=1-9c_3^2$

If $|c_3|<\frac{1}{3}$

$$\frac{y^2}{(1-9c_3^2)/2^2}+\frac{z}{(1-9c_3^2)}=1$$



(29) $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$

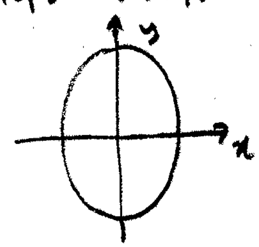


Set $z=c_1$

$\frac{x^2}{4}+\frac{y^2}{9}-c_1^2=1$

$\frac{x^2}{4}+\frac{y^2}{9}=1+c_1^2$

$$\frac{x^2}{4(1+c_1^2)}+\frac{y^2}{9(1+c_1^2)}=1$$



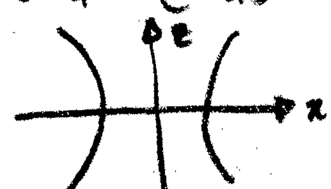
Set $y=c_2$

$\frac{x^2}{4}+\frac{c_2^2}{9}-z^2=1$

$\frac{x^2}{4}-z^2=1-\frac{c_2^2}{9}$

If $|c_2|<3$

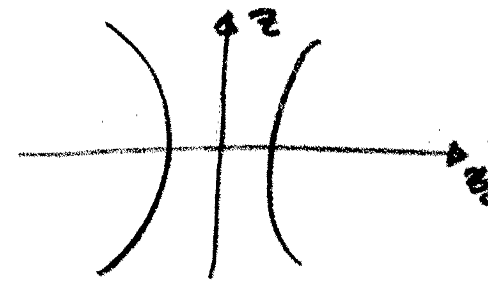
$$\frac{x^2}{4(1-\frac{c_2^2}{9})}-\frac{z^2}{(1-\frac{c_2^2}{9})}=1$$



Set $x=c_3$

similarly

$$\frac{y^2}{9(1-\frac{c_3^2}{4})}-\frac{z^2}{(1-\frac{c_3^2}{4})}=1$$



11.2

(3) $\lim_{(x,y)\to(1,0)} (xy^2 + x^3y + 5) = (1)(0)^2 + (1)^3(0) + 5 = 5$

(Since it in a polynomial & it is known to be continuous)

(5) $\lim_{(x,y)\to(1,3)} \dfrac{x+y}{x-y} = \dfrac{1+3}{1-3} = \dfrac{4}{-2} = -2$

(Ratio of polynomials, which is continuous when the denominator is not zero)

(7) $\lim_{(x,y)\to(1,0)} e^{xy} = e^{(1)(0)} = e^0 = 1$

(Since the exponential function is continuous)

(9) $\lim_{(x,y)\to(0,1)} e^{x^2+x} \ln(ey^2) = e^{0^2+0} \ln[(e)(1)] = 1$

(By continuity of exponential and logarithm as functions of one variable)

(11) $\lim_{(x,y)\to(0,0)} \underbrace{\dfrac{x^2-2xy+y^2}{x-y}}_{0/0 \text{ form}} = \lim_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{(x-y)} = \lim_{(x,y)\to(0,0)} (x-y) = 0$

(13) $\lim_{(x,y)\to(0,0)} \dfrac{e^x \tan^{-1}(y)}{y} = \left[\lim_{x\to 0} e^x\right] \left[\lim_{y\to 0} \underbrace{\dfrac{\tan^{-1} y}{y}}_{\substack{0/0 \text{ form} \\ \Rightarrow \text{L'Hôpital}}}\right]$

$= (1)(1)$

(exponential is continuous and, L'Hôpital...)

(15) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{(x+y)} = 1$ (just like $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} = 1$)

---

(17) $\displaystyle \lim_{(x,y)\to(5,5)} \frac{x^4-y^4}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{(x^2-y^2)} = \lim_{(x,y)\to(0,0)} x^2+y^2$

$= 50$

---

(19) $\displaystyle \lim_{(x,y)\to(2,1)} \frac{x^2-4y^2}{x-2y} = \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x+2y)}{(x-2y)} = \lim_{(x,y)\to(2,1)} x+2y$

$= 4$

---

(21) $\displaystyle \lim_{(x,y)\to(2,1)} (xy^2+x^3y) = (2)(1)^2+(2)^3(1) = 10$

(Limit exists by the continuity of polynomials)

---

(23) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$

If we set $x=0$,

$\displaystyle \lim_{y\to 0} \frac{0+y}{0-y} = \lim_{y\to 0} -1 = -1$

If we set $y=0$

$\displaystyle \lim_{x\to 0} \frac{x+0}{x-0} = \lim_{x\to 0} 1 = 1$

Not the same

Limit does not exist.

(25) $\lim_{(x,y)\to(0,0)} e^{xy} = e^{(0)(0)} = 1$ (exponential is continuous)

---

(27) $\lim_{(x,y)\to(0,0)} (\sin x - \cos y) = \sin(0) - \cos(0) = 0 - 1 = -1$

(Sin & cos are continuous)

---

(29) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{1-\cos(x^2+y^2)}{x^2+y^2}}_{0/0 \text{ form}} = 0$ from the graph

From the graph it is clear that, close to the origin, the function $\frac{1-\cos(x^2+y^2)}{x^2+y^2}$ takes values close to 0.

---

(31) $f(x,y) = \frac{x-y^2}{x^2+y^2}$.

First set $x=0 \Rightarrow f(0,y) = \frac{-y^2}{y^2} = -1$

$\lim_{y\to 0} f(0,y) = -1$

Then, set $y=0 \Rightarrow f(x,0) = \frac{x}{x^2} = \frac{1}{x}$

$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{1}{x} = \infty$

These two are different, so, $\lim_{(x,y)\to 0} f(x,y)$ does not exist.

30 $f(x,y) = \dfrac{x^4y^4}{(x^2+y^4)^3}$

First Set $x=0$, $f(0,y)=0$

So, $\lim_{y\to 0} f(0,y) = 0$

Then set $x=y^2$, $f(y^2,y) = \dfrac{(y^2)^4 y^4}{((y^2)^2+y^4)^3} = \dfrac{y^{12}}{(2y^4)^3} = \dfrac{1}{8}$

So, $\lim_{y\to 0} f(y^2,y) = \dfrac{1}{8}$

These two are different, so, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

---

33 $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$

First Set $x=0$ $f(0,y)=0$, so, $\lim_{y\to 0} f(0,y)=0$

Then, set $x=y$, then, $f(y,y) = \dfrac{y^4}{y^4+y^4} = \dfrac{1}{2}$

So, $\lim_{y\to 0} f(y,y) = \dfrac{1}{2}$

These two are different.

ALT. you could just set $y=mx$.

$f(x,mx) = \dfrac{x^2(mx)^2}{x^4+(mx)^4} = \dfrac{m^2x^4}{x^4(1+m^4)} = \dfrac{m^2}{1+m^4}$

So, $\lim_{x\to 0} f(x,mx) = \dfrac{m^2}{1+m^4}$, which depends on m, and will take different values for each m)

35 $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

Note that for $x \neq y$, and $x = y^3$

$$f(y^3, y) = \frac{y^4}{y^6+y^6} = \frac{1}{2y^2}$$

and $\lim_{y \to 0} f(y^3, y) = \lim_{y \to 0} \frac{1}{2y^2} = \infty$

but for, $x \neq y$, $y = 0$

$f(x,0) = 0$ and $\lim_{x \to 0} f(x,0) = 0$

$\therefore \lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

$\therefore f(x,y)$ Cannot be continuous at $(0,0)$

37

$$f(x,y) = \frac{x^2+2y^2}{x^2+y^2} \quad \text{for} \quad (x,y) \neq (0,0)$$

$$\lim_{(x,y)\to(3,1)} f(x,y) = \frac{3^2+(2)(1)^2}{3^2+(2)^2} = \frac{9+2}{9+4} = \frac{11}{13}$$

(Because $f(x,y)$, for $(x,y) \neq 0$, is a ratio of two polynomials with the denominator not equal to zero, and with $x=3$ and $y=1$, the denominator is clearly not zero)

When $y = mx$,

$$f(x,mx) = \frac{x^2+2m^2x^2}{x^2+m^2x^2} = \frac{1+2m^2}{1+m^2}$$

So, $\lim_{x\to 0} f(x,mx) = \frac{1+2m^2}{1+m^2}$, and the limit depends on m,

So, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

39 $f(x,y) = \begin{cases} \dfrac{3x^3 - 3y^3}{x^2 - y^2} & \text{for } x^2 \neq y^2 \\ B & \end{cases}$

for $x^2 \neq y^2$, $f(x,y) = \dfrac{3(x^3 - y^3)}{(x^2 - y^2)}$

$$= \frac{3(x-y)(x^2+xy+y^2)}{(x-y)(x+y)}$$

$$f(x,y) = \frac{3(x^2+xy+y^2)}{(x+y)}$$

Since we are told that the limit exists at (0,0), we can set $y = 0$,

$$f(x,0) = \frac{3x^2}{x} = 3x \text{ for } x \neq 0$$

$$\lim_{x \to 0} f(x,0) = 0 = \lim_{(x,y)\to(0,0)} f(x,y)$$

SINCE WE ARE TOLD THAT THE LIMIT EXISTS!

Hence $B = 0$.

11.3

③ $f(x,y) = x^3 + x^2y + xy^2 + y^3$

$f_x = 3x^2 + 2xy + y^2$

$f_y = x^2 + 2xy + 3y^2$

$f_{xx} = 6x + 2y$

$f_{yx} = 2x + 2y.$

⑤ $f(x,y) = \frac{x}{y}$

$f_x = \frac{1}{y}$

$f_y = \frac{-x}{y^2}$

$f_{xx} = 0$

$f_{yx} = -\frac{1}{y^2}$

⑦ $f(x,y) = \ln(2x+3y)$

$f_x = \frac{2}{2x+3y}$

$f_y = \frac{3}{2x+3y}$

$f_{xx} = \frac{-4}{(2x+3y)^2}$

$f_{yx} = \frac{-6}{2x+3y}$

(9) (a) $f(x,y) = (\sin x^2)(\cos y)$

$f_x = (2x)(\cos x^2)(\cos y)$

$f_y = -(\sin x^2)(\sin y)$

(b) $f(x,y) = \sin(x^2 \cos y)$

$f_x = 2x(\cos y)(\sin(x^2 \cos y))$

$f_y = \cos(x^2 \cos y)\, x^2 \cdot (-\sin y)$

$= -x^2 (\sin y)(\cos(x^2 \cos y))$

(11) (a)

$f(x,y) = \sqrt{3x^2+y^4}$

$f_x = \frac{1}{2} \cdot \frac{1}{\sqrt{3x^2+y^4}} \cdot 6x$

$f_y = \frac{1}{2} \frac{1}{\sqrt{3x^2+y^4}} \cdot 4y^3$

(13) $f(x,y) = x^2 e^{x+y} \cos y$

$f_x = 2x e^{x+y} \cos y + x^2 e^{x+y} \cos y$

$f_y = x^2 e^{x+y} \cos y - x^2 e^{x+y} \sin y$

(15) $f(x,y) = \sin^{-1}(xy)$

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$f_x = \frac{1}{\sqrt{1-(xy)^2}} \cdot y$

$f_y = \frac{1}{\sqrt{1-(xy)^2}} \cdot x$

(17) $f(x,y,z) = xy^2 + yz^3 + xyz$

$f_x = y^2 + 0 + yz = y^2 + yz$

$f_y = 2xy + z^3 + xz$

$f_z = 0 + 3yz^2 + xy = 3yz^2 + xy$

---

(19) $f(x,y,z) = \dfrac{x+y^2}{z} = \dfrac{x}{z} + \dfrac{y^2}{z}$

$f_x = \dfrac{1}{z}$; $f_y = \dfrac{2y}{z}$; $f_z = -\dfrac{(x+y^2)}{z^2}$

---

(21) $f(x,y,z) = \ln(x+y^2+z^3)$

$f_x = \dfrac{1}{x+y^2+z^3}$; $f_y = \dfrac{2y}{x+y^2+z^3}$; $f_z = \dfrac{3z^2}{x+y^2+z^3}$

---

(23) $\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{2} = 1$

Assumption: x & y are independent and z is a function of x & y, So, $\dfrac{\partial y}{\partial x} = 0$, $\dfrac{\partial x}{\partial y} = 0$

Partially differentiate w.r.t. x.

$\dfrac{2x}{9} - 0 + \dfrac{2z}{2}\dfrac{\partial z}{\partial x} = 0 \Rightarrow \dfrac{\partial z}{\partial x} = -\dfrac{2x}{9z}$

Partially differentiate w.r.t y

$0 - \dfrac{2y}{4} + \dfrac{2z}{2}\dfrac{\partial z}{\partial y} = 0 \Rightarrow \dfrac{\partial z}{\partial y} = \dfrac{y}{2z}$

(27) $\sqrt{x} + y^2 + \sin xz = 2$

Partially in $x \Rightarrow$

$$\frac{\partial \sqrt{x}}{\partial x} + \cancel{\frac{\partial y^2}{\partial x}} + \frac{\partial}{\partial x}\sin xz = \cancel{\frac{\partial 2}{\partial x}}$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{x}} + (\cos xz)\frac{\partial (xz)}{\partial x} = 0$$

$$\frac{1}{2\sqrt{x}} + (\cos xz)\left(z + x\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{1}{2\sqrt{x}} + z\cos xz + x(\cos xz)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x} - \frac{1}{2x\sqrt{x}\cos xz}$$

Partially in y:

$$\cancel{\frac{\partial \sqrt{x}}{\partial y}} + \frac{\partial y^2}{\partial y} + \frac{\partial}{\partial y}\sin xz = \cancel{\frac{\partial 2}{\partial y}}$$

$$2y + x(\cos xz)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{2y}{x\cos xz}$$

(25) $3x^2y + y^3z - z^2 = 1$

Partially differentiate in x.

$$6xy + y^3 \frac{\partial z}{\partial x} - 2z\frac{\partial z}{\partial x} = 0$$

$$6xy + (y^3 - 2z)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-6xy}{y^3 - 2z} = \frac{6xy}{2z - y^3}.$$

Partially differentiate in y

$$\frac{\partial(3x^2y)}{\partial y} + \underbrace{\frac{\partial(y^3 z)}{\partial y}}_{\text{product rule}} - \frac{\partial z^2}{\partial y} = \frac{\partial 1}{\partial y}$$

$$3x^2 + 3y^2 z + y^3\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0$$

$$3(x^2 + y^2 z) + (y^3 - 2z)\frac{\partial z}{\partial y} = 0$$

So, $$\frac{\partial z}{\partial y} = \frac{-3(x^2 + y^2 z)}{y^3 - 2z}$$

Slope of the tangent parallel to:

xz plane $\longrightarrow \dfrac{\partial z}{\partial x}$

yz plane $\longrightarrow \dfrac{\partial z}{\partial y}$

(29) $z = f(x,y) = xy^3 + x^3y \quad ; P_0(1, -1, -2)$

(a) $f_x = \dfrac{\partial f}{\partial x} = y^3 + 3x^2y \Rightarrow f_x(1,-1) = (-1)^3 + (3)(1)(-1)$
$= -4$

$\therefore$ Slope of the tangent parallel to the xz plane is -4.

(b) $f_y = \dfrac{\partial f}{\partial y} = 3y^2x + x^3 \Rightarrow f_y(1,-1) = 3(-1)^2(1) + 1^3$
$= 4$

Slope of the tangent parallel to the yz plane is 4

31 $f(x,y) = x^2 \sin(x+y)$

$f_x = 2x\sin(x+y) + x^2\cos(x+y)$

$$f_x\left(\frac{\pi}{2},\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right)\underbrace{\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}_{=0} + \left(\frac{\pi}{2}\right)^2\underbrace{\cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}_{=-1}$$

$$= -\frac{\pi^2}{4}$$ ← slope of tangent parallel to xz plane

$\cancel{f_y = x^2\cos(x+y)}$

$$f_y\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2\underbrace{\cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}_{=-1} = -\frac{\pi^2}{4}$$

slope of tangent parallel to yz plane.

11.5

(5) $f(x,y) = (4+y^2)x$, $x = e^{2t}$, $y = e^{3t}$
$= 4x + y^2 x$

(a) Let $F(t) = f(e^{2t}, e^{3t}) = (4+e^{6t})e^{2t}$
$= 4e^{2t} + e^{8t}$

$$\frac{df}{dt} = F'(t) = 8e^{2t} + 8e^{8t}$$

(b) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (4+y^2)\cdot(2e^{2t}) + (2xy)(3e^{3t})$$

$$= (4+e^{6t})(2e^{2t}) + (2e^{2t}\cdot e^{3t})(3e^{3t})$$

$$= 8e^{2t} + 2e^{8t} + 6e^{8t}$$

$$\frac{df}{dt} = 8e^{2t} + 8e^{8t}$$

(7) $f(x,y) = xy$, $x = \cos 3t$, $y = \tan 3t$

(a) So, $f(\cos 3t, \tan 3t) = \cos 3t \cdot \tan 3t$

$$= \cos 3t \cdot \frac{\sin 3t}{\cos 3t}$$

$$= \sin 3t.$$

$$\therefore \frac{df}{dt} = 3\cos 3t$$

(b) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (y)(-3\sin 3t) + (x)(3\sec^2 3t)$$

$$= -3\left(\frac{\sin 3t}{\cos 3t}\right)\sin 3t + 3(\cos 3t)\left(\frac{1}{\cos^2 3t}\right)$$

$$= -3\,\frac{\sin^2 3t}{\cos 3t} + 3 \cdot \frac{1}{\cos 3t}$$

$$= \frac{3(1-\sin^2 3t)}{\cos 3t} = \frac{3(\cos^2 3t)}{\cos 3t}$$

$$\frac{df}{dt} = 3\cos 3t$$

⑨ $F(x,y) = x^2 + y^2$; $x = u\sin v$
$y = u - 2v$

(a) $f(u\sin v, u-2v) = u^2\sin^2 v + (u-2v)^2$

$= u^2\sin^2 v + u^2 - 4uv + 4v^2$

$$\frac{\partial f}{\partial u} = 2u\sin^2 v + 2u - 4v$$

$$\frac{\partial F}{\partial v} = 2u^2\sin v\cos v - 4u + 8v$$

(b) $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$= (2x)(\sin v) + (2y)(1)$$

$$\frac{\partial f}{\partial u} = 2u\sin^2 v + 2u - 4v$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= (2x)(u\cos v) + (2y)(-2)$$

$$\frac{\partial F}{\partial v} = 2u\sin v\cos v - 4u - 8v$$

(11) $F(x,y) = \ln xy \quad , \quad x = e^{uv^2}, y = e^{uv}$

$$f(e^{uv^2}, e^{uv}) = \ln\left(e^{uv^2+uv}\right) = uv^2+uv$$

$$\frac{\partial F}{\partial u} = v^2+v \quad ; \quad \frac{\partial F}{\partial v} = 2uv+u$$

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}$$

$$= \left(\frac{y}{xy}\right)\left(v^2e^{uv^2}\right) + \left(\frac{x}{xy}\right)\left(ve^{uv}\right)$$

$$= \left(\frac{1}{x}\right)\cdot v^2\underbrace{e^{uv^2}}_{=x} + \left(\frac{1}{y}\right)v\underbrace{e^{uv}}_{=y}$$

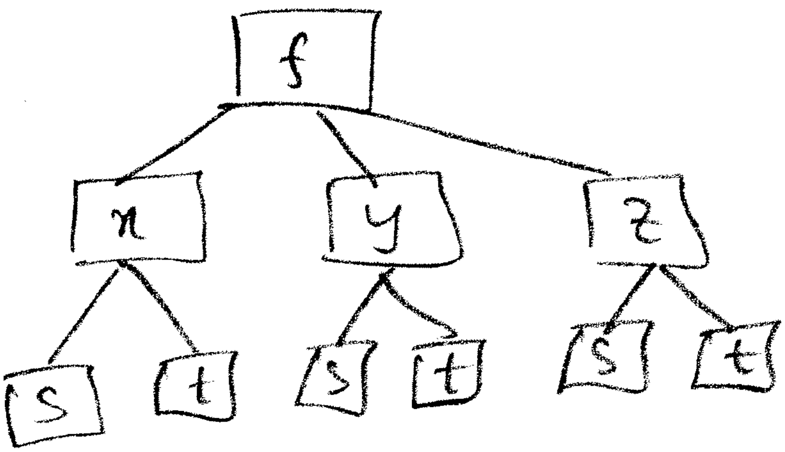
$$\frac{\partial F}{\partial u} = v^2+v$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v}$$

$$= \left(\frac{y}{xy}\right)\left(2vu\underbrace{e^{uv^2}}_{=y}\right) + \left(\frac{x}{xy}\right)\left(u\underbrace{e^{uv}}_{=x}\right)$$

$$\boxed{\frac{\partial F}{\partial v} = 2uv+u}$$

(13) $w = f(x,y,z)$; $x = x(s,t)$
$y = y(s,t)$
$z = z(s,t)$



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

---

(15) $w = f(x,y,z)$, $x = x(s,t,u)$, $y = y(s,t,u)$, $z = z(s,t,u)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

(29) $xyz = 2 \Rightarrow z = \frac{2}{xy} \Rightarrow xy = \frac{2}{z}$

(a) $\frac{\partial}{\partial y}(xyz) = \underbrace{\frac{\partial \cancel{2}}{\partial y}}_{0}$

$$xz + xy\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{xz}{xy} = \frac{-z}{y}$$

So $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{-z}{y}\right) = -\frac{1}{y}\frac{\partial z}{\partial x} \quad \text{— (1)}$$

$$\frac{\partial}{\partial x}(xyz) = \underbrace{\frac{\partial \cancel{2}}{\partial x}}_{0} \Rightarrow yz + xy\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-z}{x} \quad \text{— (2)}$$

$\therefore$ from (1) $\Rightarrow$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{z}{xy} = \frac{z^2}{2}$$

This is the general method, but for this, we have an easier way for this problem.

$$z = \frac{2}{xy} \Rightarrow \frac{\partial z}{\partial y} = \frac{-2}{xy^2} \Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{2}{x^2y^2} = \frac{z^2}{2}$$

(b) $z = \dfrac{2}{xy}$

So, $\dfrac{\partial z}{\partial x} = \dfrac{-2}{x^2 y} \Rightarrow \dfrac{\partial^2 z}{\partial x^2} = \dfrac{4}{x^3 y} = \left(\dfrac{2}{xy}\right)\left(\dfrac{2}{x^2}\right)$

$\therefore \dfrac{\partial^2 z}{\partial x^2} = \dfrac{2z}{x^2}$

(c) $z = \dfrac{2}{xy}$

$\dfrac{\partial z}{\partial y} = \dfrac{-2}{xy^2} \Rightarrow \dfrac{\partial^2 z}{\partial y^2} = \dfrac{4}{xy^3} = \dfrac{2z}{y^2}$

---

(3) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 3$

$\Rightarrow z = \left[3 - \dfrac{1}{x} - \dfrac{1}{y}\right]^{-1}$

$$\boxed{\begin{aligned}(a)\ \frac{\partial^2 z}{\partial x \partial y} &= -2y^{-2} z \frac{\partial z}{\partial x} \\ &= -\frac{2}{y^2} \cdot z \cdot \left(\frac{-z^2}{x^2}\right) \\ &= \frac{2z^3}{x^2 y^2}\end{aligned}}$$

$\dfrac{\partial z}{\partial y} = -1\left[3 - \dfrac{1}{x} - \dfrac{1}{y}\right]^{-2}\left(-\left(\dfrac{-1}{y^2}\right)\right) = -\dfrac{z^2}{y^2} = -y^{-2} z^2$

(c) $\dfrac{\partial^2 z}{\partial y^2} = -y^{-2}(2z)\dfrac{\partial z}{\partial y} + 2y^{-3} z^2 = \dfrac{2z^3}{y^4} + \dfrac{2z^2}{y^3} = \dfrac{2z^2(z+y)}{y^4}$

$\dfrac{\partial z}{\partial x} = -1\left[3 - \dfrac{1}{x} - \dfrac{1}{y}\right]^{-2}\left(-\left(\dfrac{-1}{x^2}\right)\right] = -\dfrac{z^2}{x^2} = -x^{-2} z^2$

(b) $\dfrac{\partial^2 z}{\partial x^2} = -x^{-2}\, 2z \dfrac{\partial z}{\partial x} + 2x^{-3} z^2 = \dfrac{2z^2(x+z)}{x^4}$