§ 12.8

* Given $x = x(u,v)$; $y = y(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

* Given $x = x(u,v,w)$; $y = y(u,v,w)$; $z = z(u,v,w)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

---

(3) $x = u^2$, $y = u + v$ $\to$ $\frac{\partial x}{\partial u} = 2u$; $\frac{\partial x}{\partial v} = 0$

$\frac{\partial y}{\partial u} = 1$; $\frac{\partial y}{\partial v} = 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 0 \\ 1 & 1 \end{vmatrix}$$

$= (2u)(1) - (1)(0)$

$= 2u$

(7) $x = e^u \sin v \; ; \; y = e^u \cos v$

$$\frac{\partial x}{\partial u} = e^u \sin v \; ; \; \frac{\partial x}{\partial v} \; e^u \cos v$$

$$\frac{\partial y}{\partial u} = e^u \cos v \; ; \; \frac{\partial y}{\partial v} = -e^u \sin v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^u \sin v & e^u \cos v \\ e^u \cos v & -e^u \sin v \end{vmatrix}$$

$$= -e^{2u} \sin^2 v - e^{2u} \cos^2 v$$

$$= -e^{2u} (\underbrace{\sin^2 v + \cos^2 v}_{=1})$$

$$= -e^{2u}$$

---

(9) $x = u + v - w \; ; \; y = 2u - v + 3w \; ; \; z = -u + 2v - w$

$$\frac{\partial x}{\partial u} = 1 \; ; \; \frac{\partial x}{\partial v} = 1 \; ; \; \frac{\partial x}{\partial w} = -1$$

$$\frac{\partial y}{\partial u} = 2 \; ; \; \frac{\partial y}{\partial v} = -1 \; ; \; \frac{\partial y}{\partial w} = 3$$

$$\frac{\partial z}{\partial u} = -1 \; ; \; \frac{\partial z}{\partial v} = 2 \; ; \; \frac{\partial z}{\partial w} = -1$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & \boxed{1} & \boxed{-1} \\ \boxed{2} & \boxed{-1} & \boxed{3} \\ \boxed{-1} & \boxed{2} & \boxed{-1} \end{vmatrix}$$

$$= 1\,((-1)(-1) - (2)(3))$$

$$-1\,((2)(-1) - (-1)(3))$$

minus sign

$$+(-1)\,((2)(2) - (-1)(-1))$$

$$= (1-6) - (-2+3) - (4-1)$$

$$= -9$$

---

(11) $x = u \cos v \; ; \; y = u \sin v \; ; \; z = w e^{uv}$

$$\frac{\partial x}{\partial u} = \cos v \; ; \; \frac{\partial x}{\partial v} = \sin v \; ; \; \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = -u \sin v \; ; \; \frac{\partial y}{\partial v} = u \cos v \; ; \; \frac{\partial y}{\partial w} = 0$$

$$\frac{\partial z}{\partial u} = wve^{uv} \; ; \; \frac{\partial z}{\partial v} = wue^{uv} \; ; \; \frac{\partial z}{\partial w} = e^{uv}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \\ wve^{uv} & wue^{uv} & e^{uv} \end{vmatrix}$$

$$= e^{uv} (u \cos^2 v + u \sin^2 v)$$

$$= u e^{uv}$$

(13) $u = 2x - 3y$; $v = x + 4y$.

Method I Solve for x & y:

$2x - 3y = u$ — (1)
$x + 4y = v$ — (2)

(2) × 2 ⇒ $2x + 8y = 2v$ — (3)

(3) − (1) $\cancel{8y}$ $11y = 2v - u$ ⇒ $y = \frac{2}{11}v - \frac{1}{11}u$

(1) × 4 ⇒ $8x - 12y = 4u$ — (4)
(2) × 3 ⇒ $3x + 12y = 3v$ — (5)
(4) + (5) ⇒ $11x = 3v + 4u$. ⇒ $x = \frac{3}{11}v + \frac{4}{11}u$.

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= \frac{4}{11} \; ; \; \frac{\partial x}{\partial v} = \frac{3}{11} \\ \frac{\partial y}{\partial u} &= \frac{-1}{11} \; ; \; \frac{\partial y}{\partial v} = \frac{2}{11} \end{aligned}\right\} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{2}{11} \end{vmatrix}$$

$$= \frac{8}{121} + \frac{3}{121}$$
$$= \frac{11}{121} = \frac{1}{11}$$

Method 2:

$\frac{\partial u}{\partial x} = 2$, $\frac{\partial u}{\partial y} = -3$

$\frac{\partial v}{\partial x} = 1$; $\frac{\partial v}{\partial y} = 4$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

$= 8 + 3$
$= 11$

Then $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{11}$.

Since $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

(15) $u = y e^{-x}$; $v = e^{x}$ $\Rightarrow$ $x = \ln v$

Method I

Note that $\frac{1}{v} = e^{-x}$

Then, $u = y \cdot \frac{1}{v}$ $\Rightarrow$ $y = uv$

So, $\frac{\partial x}{\partial u} = 0$; $\frac{\partial x}{\partial v} = \frac{1}{v}$

$\frac{\partial y}{\partial u} = v$; $\frac{\partial y}{\partial v} = u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & \frac{1}{v} \\ v & u \end{vmatrix}$$

$= -(v)\left(\frac{1}{v}\right)$

$= 1$

Method 2

$\frac{\partial u}{\partial x} = -y e^{-x}$; $\frac{\partial u}{\partial y} = e^{-x}$

$\frac{\partial v}{\partial x} = e^{x}$; $\frac{\partial v}{\partial y} = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y e^{-x} & e^{-x} \\ e^{x} & 0 \end{vmatrix} = \cancel{(-y e^{-x})(0)}^{\,0} - \underbrace{(e^{x})(e^{-x})}_{=1}$$

$= -1$.

So, $\frac{\partial(x,y)}{\partial(u,v)} = -1$.

(17) $u = x^2 - y^2$; $v = x^2 + y^2$

Method 2.

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = 2x \;;\; \dfrac{\partial u}{\partial y} = -2y \\[2ex] \dfrac{\partial v}{\partial x} = 2x \;;\; \dfrac{\partial v}{\partial y} = 2y \end{array}\right\} \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}$$

$$= (2x)(2y) - (2x)(-2y)$$
$$= 4xy + 4xy$$
$$= 8xy.$$

So, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{8xy}$ ← $\boxed{\text{This is what is given in the book.}}$

Still we have to write x & y in term of u & v!

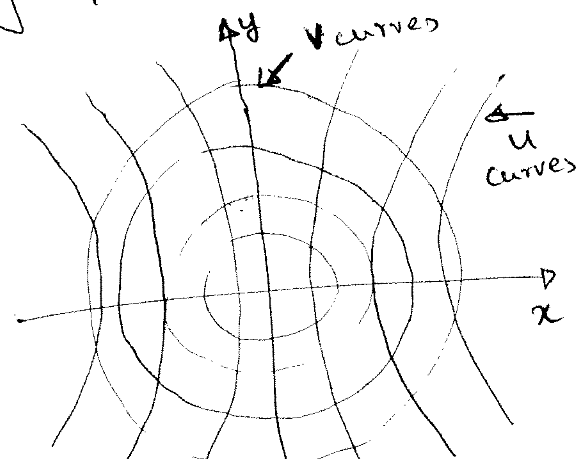
You can solve for x & y and multiply them out. But there is a better way for this problem!

$u^2 = (x^2-y^2)(x^2-y^2) = x^4 - 2x^2y^2 + y^4$

$v^2 = (x^2+y^2)(x^2+y^2) = x^4 + 2x^2y^2 + y^4$

$\Rightarrow v^2 - u^2 = 4x^2y^2 \Rightarrow$ taking square roots $2xy = \pm\sqrt{v^2-u^2}$

So, $\dfrac{\partial(x,y)}{\partial(u,v)} = \pm\dfrac{1}{4\sqrt{v^2-u^2}}$



$\boxed{\text{In the problem of changing variables of an integral you only need } \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{4\sqrt{v^2-u^2}}}$

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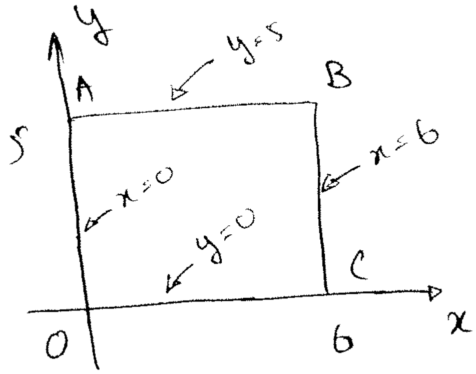
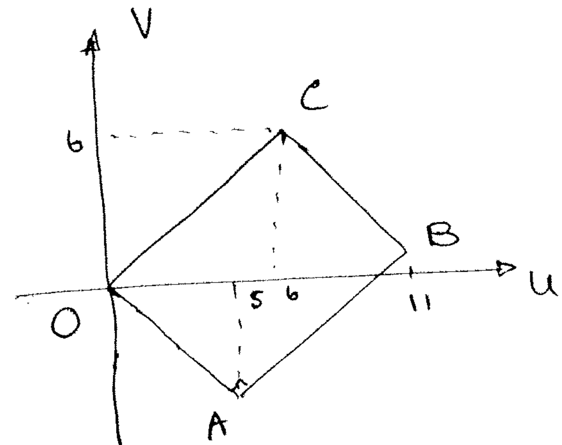




$u = x + y$

$v = x - y$

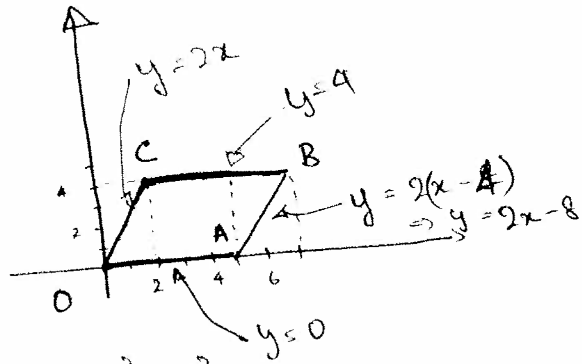
	(x,y)	(u,v)
O	$(0,0)$	$(0,0)$
A	$(0,5)$	$(5,-5)$
B	$(6,5)$	$(11,1)$
C	$(6,0)$	$(6,6)$

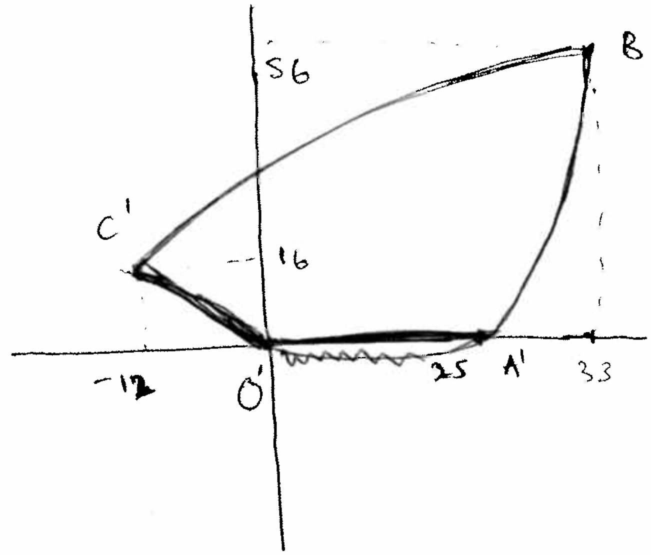
Mapping of points.

Segment	x&y Eq.	u&v Equation
OA	$x = 0$	$u = y$ & $v = -y \Rightarrow v = -u$
AB	$y = 5$	$u = x+5$ & $v = x-5$ $u - v = 10 \Rightarrow v = u-10$
BC	$x = 6$	$u = 6+y$ & $v = 6-y$ $u + v = 12 \Rightarrow v = 12-u$
CO	$y = 0$	$u = x$ & $v = x$ $\Rightarrow u = v$

Mapping of line segments

(21)





$u = x^2 - y^2$

$v = 2xy.$

Mapping of points

	(x, y)	(u, v)
O	$(0,0)$	$(0,0)$
A	$(5,0)$	$(25, 0)$
B	$(7,4)$	$(33, 56)$
C	$(2,4)$	$(-12, 16)$

49
16
33

4
16

Mapping of line segments

Segment	x & y Equation	u & v Equation
(OA)	$y = 0$	$u = x^2$; $v = 0$.
(AB)	$y = 2x - 8$	$u = x^2 - (2(x-4))^2$ $u = x^2 - 4(x^2 - 8x + 16)$ $u = -3x^2 + 32x - 64$ $v = 4x(x-4)$
(BC)	$y = 4$	$\left.\begin{array}{l} x^2 - 16 = u \\ v = 8x \end{array}\right\}$ $u = \left(\frac{v}{8}\right)^2 - 16$
(CO)	$y = 2x$	$u = x^2 - 4x^2 = -3x^2$ $v = (2x)(2x) = 4x^2$ $4u = -3v$

(22)

$$\boxed{dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

(23)

$$x = u(1-v) \quad ; \quad y = uv$$

$$\frac{\partial x}{\partial u} = (1-v) \quad ; \quad \frac{\partial x}{\partial v} = -u$$

$$\frac{\partial y}{\partial u} = v \quad ; \quad \frac{\partial y}{\partial v} = u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) - (v)(-u)$$

$$= u - uv + uv$$

$$= u$$

$$\therefore \; dx\,dy = u\,du\,dv$$

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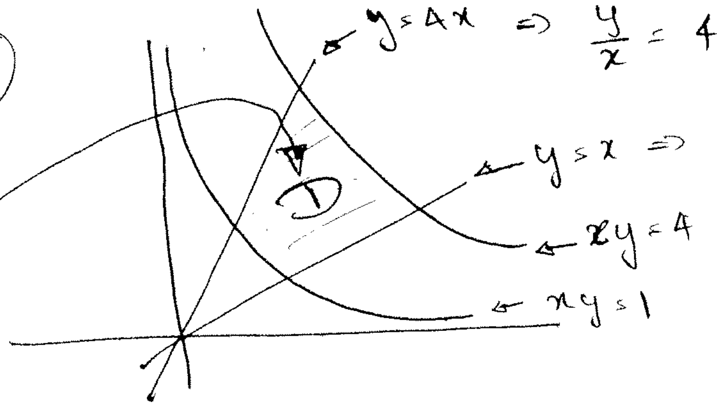


Let

$$u = \frac{y}{x} \quad \& \quad v = xy$$

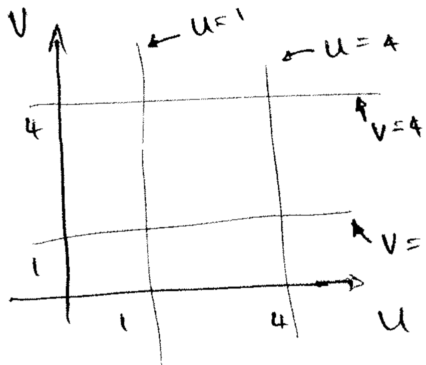
$$\frac{\partial u}{\partial x} = \frac{-y}{x^2} \; ; \; \frac{\partial u}{\partial y} = \frac{1}{x} \; ; \; \frac{\partial v}{\partial x} = y \; ; \; \frac{\partial v}{\partial y} = x.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = \left(\frac{-y}{x^2}\right)(x) - \left(\frac{1}{x}\cdot y\right)$$

$$= -\frac{y}{x} - \frac{y}{x} = -\frac{2y}{x}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -2u$$

$$\text{So, } \frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{2u}$$



$$\text{Area} = \iint_D dx\,dy = \int_1^4 \int_1^4 \left|\frac{-1}{2u}\right| du\,dv$$

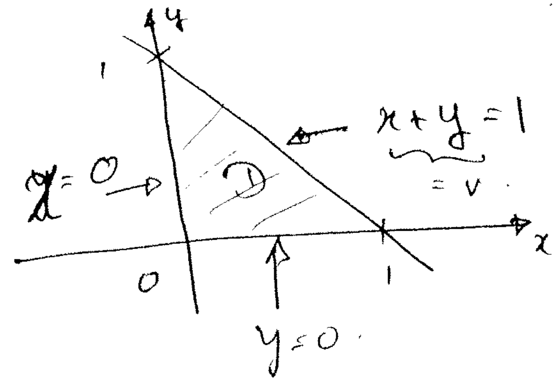
$$= \int_1^4 \int_1^4 \frac{1}{u}\,du\,dv = \frac{1}{2}\int_1^4 \left[\ln|u|\right]_1^4 dv$$

$$= \frac{1}{2}\left[\ln|u|\right]_1^4 \left[v\right]_1^4 = \frac{1}{2}(\ln 4 - \ln 1)(4-1)$$

$$\boxed{\text{Area of } D = \frac{3}{2}\ln 4 = 3\ln 2}$$

$$\frac{1}{2}\ln 4 = \ln\sqrt{4} = \ln 2$$

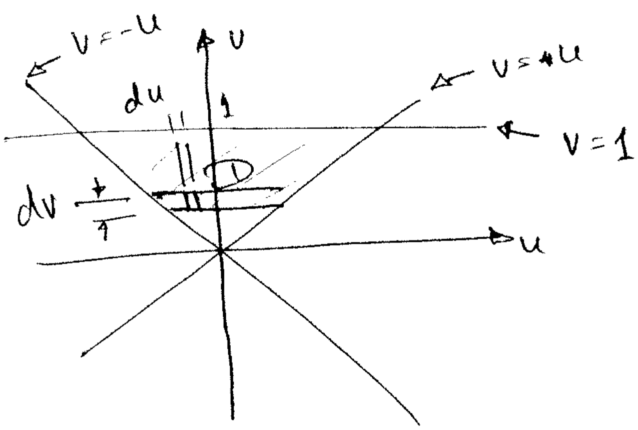
(29) $\iint_D (x-y)^5 (x+y)^3 \, dy\,dx$



$u = x - y, \quad v = x + y.$

$u + v = 2x \Rightarrow$ when $x = 0 \Rightarrow u + v = 0 \Rightarrow v = -u$

$v - u = 2y \Rightarrow$ when $y = 0 \Rightarrow v - u = 0 \Rightarrow v = u.$



$\frac{\partial u}{\partial x} = 1 \; ; \; \frac{\partial u}{\partial y} = -1$

$\frac{\partial v}{\partial x} = 1 \; ; \; \frac{\partial v}{\partial y} = 1$

$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 - (-1) = 2$

So, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$

$$\iint_D (x-y)^5 (x+y)^3 \, dy\,dx = \int_0^1 \int_{-v}^{v} u^5 v^3 \, du\,dv$$

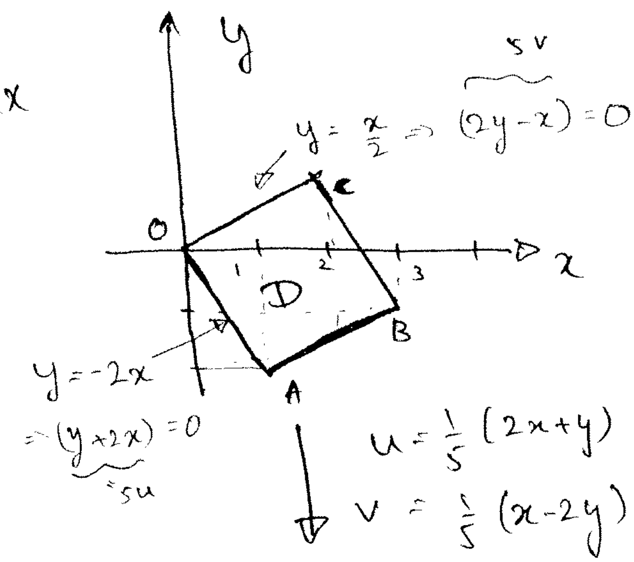
$$= \int_0^1 \left[\frac{u^6}{6} v^3 \right]_{-v}^{v} dv$$

$$= \int_0^1 v^3 \left[\frac{v^6}{6} - \frac{(-v)^6}{6} \right] dv$$

$$= \int_0^1 v^3 \left[\underbrace{\frac{v^6}{6} - \frac{v^6}{6}}_{=0} \right] dv$$

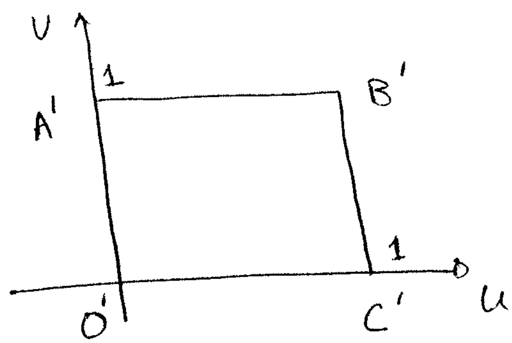
$$= 0$$

(35) $\iint_D (2x+y)(x-2y)^2\,dy\,dx$



$\frac{\partial u}{\partial x} = \frac{2}{5}$; $\frac{\partial u}{\partial y} = \frac{1}{5}$

$\frac{\partial v}{\partial x} = \frac{1}{5}$; $\frac{\partial v}{\partial y} = -\frac{2}{5}$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{vmatrix}$$

$$= \left(\frac{2}{5}\right)\left(\frac{-2}{5}\right) - \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$$

$$= -\frac{4}{25} - \frac{1}{5} = \frac{-5}{25} = \frac{-1}{5}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = -5$$

$$\iint_D (2x+y)(x-2y)^2\,dy\,dx = \int_0^1\int_0^1 5u\,(5v)^2 \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=|-5|} dv\,du$$

$$= 5^4 \int_0^1\int_0^1 uv^2\,dv\,du$$

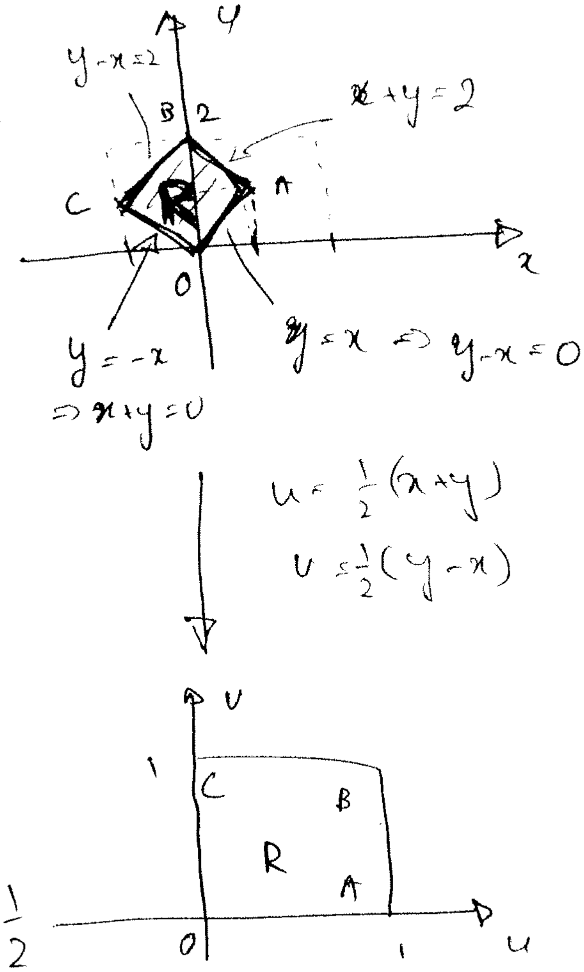
$$= 5^4 \int_0^{1} \left[\frac{uv^3}{3}\right]_{v=0}^{v=1} du$$

$$= 5^4 \int_0^1 \frac{u}{3}\,du$$

$$= 5^4 \left[\frac{u^2}{6}\right]_0^1$$

$$= \frac{5^4}{6} = \frac{625}{6}$$

(41) $\displaystyle\iint_R \left(\frac{x+y}{2}\right)^2 e^{(y-x)/2}\, dy\, dx$



Let $u = \frac{1}{2}(x+y)$

$v = \frac{1}{2}(y-x)$

Then, $\dfrac{\partial u}{\partial x} = \dfrac{1}{2}$; $\dfrac{\partial u}{\partial y} = \dfrac{1}{2}$

$\dfrac{\partial v}{\partial x} = -\dfrac{1}{2}$; $\dfrac{\partial v}{\partial y} = \dfrac{1}{2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = 2.$$

$$\iint_R \left(\frac{x+y}{2}\right)^2 e^{(y-x)/2}\, dy\, dx = \int_0^1\!\!\int_0^1 u^2 e^v \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=2} du\, dv$$

$$= \int_0^1 2\left[u^2 e^v\right]_0^1 du = \int_0^1 2u^2 \left[e^v\right]_0^1 du$$

$$= 2\left[\frac{u^3}{3}\right]_0^1 \left[e^v\right]_0^1$$

$$= \frac{2}{3}\left[e^1 - 1\right]$$

(53) $5x^2 - 4xy + 2y^2 = 1$; $u = x+2y$; $v = 2x - y$.

set $5x^2 - 4xy + 2y^2 = A(x+2y)^2 + B(2x-y)^2$

$= A(x^2 + 4xy + 4y^2) + B(4x^2 - 4xy + y^2)$

$= (A+4B)x^2 + (4A-4B)xy + (4A+B)y^2$

$\Rightarrow A + 4B = 5$ — (1)

$4A + B = 2$ — (2)

(1) $\times 4 \Rightarrow 4A + 16B = 20$ — (3)

(2) $\Rightarrow 4A + B = 2$

(3) - (2) $15B = 18 \Rightarrow B = \frac{6}{5}$.

(2) $\times 4 \Rightarrow 16A + 4B = 8$ — (4)

$A + 4B = 5$

(4) - (1) $15A = 3$

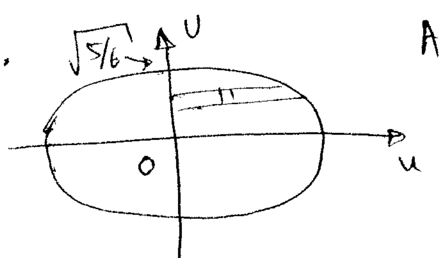
$\Rightarrow A = \frac{1}{5}$.

Check $4A - 4B = 4(A-B) = 4\cdot\left(\frac{1}{5} - \frac{6}{5}\right) = -4$

This matches with the coefficient of xy.

So, the ellipse can be written as

$$\frac{(x+2y)^2}{5} + \frac{6(2x-y)}{5} = 1 \Rightarrow \frac{u^2}{5} + \frac{6v^2}{5} = 1 \Rightarrow u^2 = 5 - 6v^2$$



Area $= 4\int_0^{\sqrt{5/6}}\int_0^{\sqrt{5-6v^2}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$

$= 4\int_0^{\sqrt{5/6}}\int_0^{\sqrt{5-6v^2}} 5\, du\, dv$

$= 20\int_0^{\sqrt{5/6}} \sqrt{5-6v^2}\, dv$

$= 20\int_0^{\sqrt{5/6}} \sqrt{5}\sqrt{\left(1 - \frac{6v^2}{5}\right)}\, dv$

$= \frac{\pi\sqrt{6}}{6}$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial u}{\partial y} = 2$

$\frac{\partial v}{\partial x} = 2$; $\frac{\partial v}{\partial y} = -1$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= -1 - 4$

$= -5$

Let $\sqrt{\frac{6}{5}}\, v = \sin\theta \Rightarrow \sqrt{\frac{6}{5}}\, dv = d\theta$

$v = \sqrt{\frac{5}{6}} \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$v = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$20\cdot\int_0^{\pi/2} \sqrt{5}\cdot\sqrt{\frac{5}{6}} \cos^2\theta\, d\theta \ldots$

§ 12.3

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta.$$

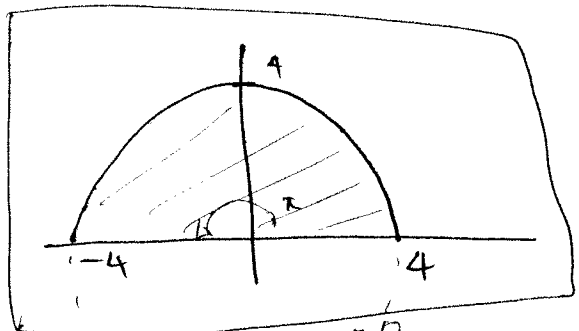
$$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

(3) $\displaystyle\int_0^{\pi}\int_0^{4} r^2\sin^2\theta \, dr\, d\theta$

$$= \int_0^{\pi}\int_0^{4} r^2\left[\frac{1-\cos 2\theta}{2}\right] dr\, d\theta$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\left(\frac{1-\cos 2\theta}{2}\right)\right]_{r=0}^{r=4} d\theta$$



4
16
64

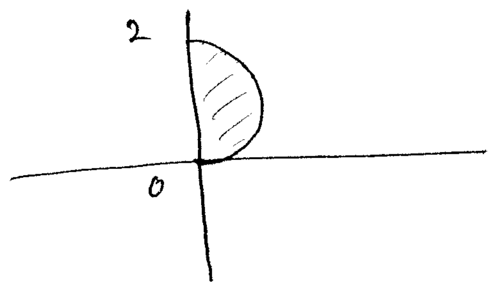
$$= \int_0^{\pi}\frac{64}{3}\left(\frac{1-\cos 2\theta}{2}\right) d\theta.$$

$$= \frac{32}{3}\int_0^{\pi} 1-\cos 2\theta \, d\theta$$

$$= \frac{32}{3}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{32}{3}\left[\left(\pi - \frac{\cancel{\sin 2\pi}}{2}\right)-\left(0-\frac{\cancel{\sin 0}}{2}\right)\right]$$

$$= \frac{32\pi}{3}$$

(7) $\displaystyle\int_0^{\pi/2}\int_0^{2\sin\theta} dr\, d\theta$



$$\int_0^{\pi/2}(2\sin\theta - 0)\, d\theta$$

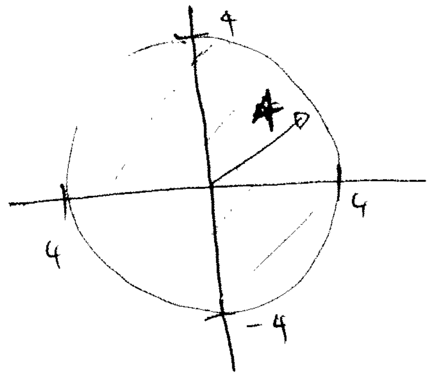
$$= \int_0^{\pi/2} 2\sin\theta \, d\theta.$$

$$= \left[-2\cos\theta\right]_0^{\pi/2}$$

$$= \left(-2\underbrace{\cancel{\cos\tfrac{\pi}{2}}}_{0}\right) - \left(-2\underbrace{\cos 0}_{=1}\right)$$

$$= 2.$$

(9)



$$\int_0^{2\pi}\int_0^{4} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^4 d\theta$$

$$= \left[\frac{r^2}{2}\right]_0^4 \left[\theta\right]_0^{2\pi}$$

$$= \left[\frac{4^2}{2} - \frac{0}{2}\right]\left[2\pi - 0\right]$$

$$= 16\pi$$

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}$

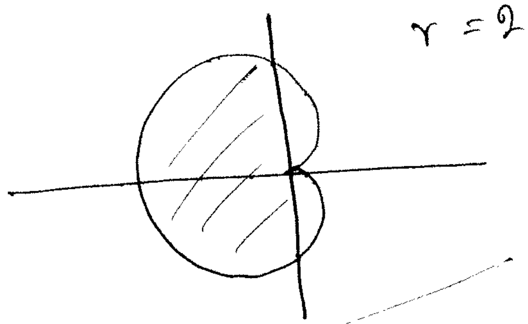
$= r\cos^2\theta + r\sin^2\theta$

$= r$

---

(11)

$r = 2(1-\cos\theta)$



$$\int_0^{2\pi}\int_0^{2(1-\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{2(1-\cos\theta)} d\theta$$

$$= \int_0^{2\pi} \frac{\cancel{4}^{2}(1-\cos\theta)^2}{\cancel{2}}\,d\theta = 2\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta$$

$$= \int_0^{2\pi} 2 - 4\cos\theta + 2\cos^2\theta\,d\theta = \int_0^{2\pi} 2 - 4\cos\theta + (1+\cos 2\theta)\,d\theta$$

$$= \int_0^{2\pi} 3 - 4\cos\theta + \cos 2\theta\,d\theta$$

$$= \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \left[6\pi - 4\sin 2\pi + \frac{\sin 4\pi}{2}\right] - \left[0 - 0 - 0\right]$$

$$= 6\pi$$

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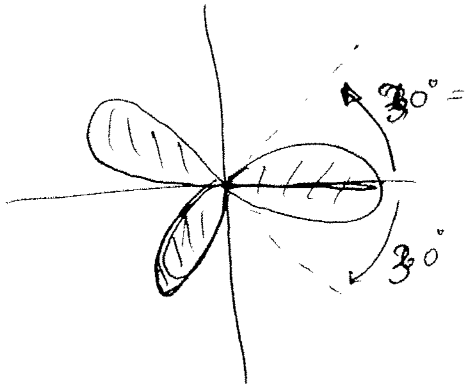

Area $= 6\int_0^{\pi/6}\int_0^{4\cos 3\theta} r\,dr\,d\theta$

$= 6\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^{4\cos 3\theta} d\theta$

By symmetry.

$= \frac{1}{2}\cdot 6\int_0^{\pi/6} 16\cos^2 3\theta\,d\theta$

$= 3\int_0^{\pi/6} 8(1+\cos 6\theta)\,d\theta$

$= 24\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta$

$= 24\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$

$= 24\left[\left(\frac{\pi}{6} + \frac{\sin \pi}{6}\right) - \left(0 + \frac{\sin 0}{6}\right)\right]$

$= \frac{24\cdot\pi}{6}$

$= 4\pi$

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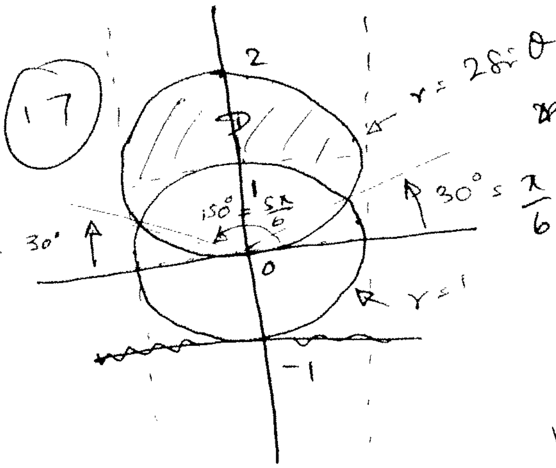


$r = 1 = 2\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°, 150°$

$$\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\right]_1^{2\sin\theta} d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{4\sin^2\theta}{2} - \frac{1}{2}\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 2\sin^2\theta - \frac{1}{2}\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 1 - \cos 2\theta - \frac{1}{2}\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{1}{2} - \cos 2\theta\,d\theta$$

$$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{5\pi}{6} - \sin\left(\frac{5\pi}{3}\right)\right) - \left(\frac{\pi}{6} - \sin\frac{\pi}{3}\right)\right]$$

$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$$= \frac{1}{2}\left[\frac{4\pi}{6} - \left(\frac{-\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{2\pi}{6} + \frac{1}{2}\cdot\frac{2\sqrt{3}}{2}$$

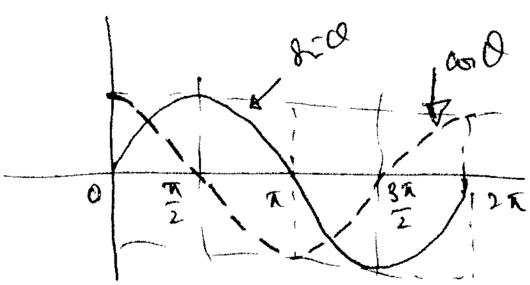
$$= \frac{1}{6}\left(2\pi + 3\sqrt{3}\right)$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

~~$\sin\frac{\pi}{3}$~~ =

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞



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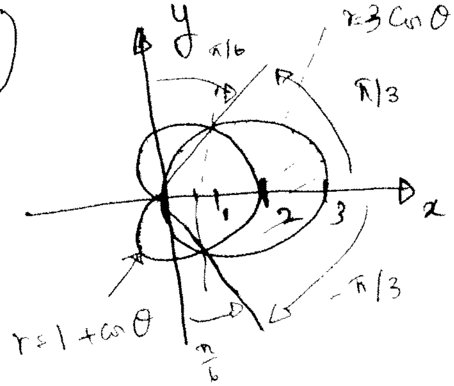


$$3\cos\theta = 1+\cos\theta$$

$$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}.$$

$$\int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_{1+\cos\theta}^{3\cos\theta} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{9\cos^2\theta}{2} - \frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$2\cos^2\theta = \cos 2\theta + 1$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 9\cos^2\theta - (1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 8\cos^2\theta - 1 - 2\cos\theta\,d\theta.$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4(\cos 2\theta + 1) - 1 - 2\cos\theta\,d\theta.$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos 2\theta + 3 - 2\cos\theta\,d\theta$$

$$= \frac{1}{2}\left[\frac{4\sin 2\theta}{2} + 3\theta - 2\sin\theta\right]_{-\pi/3}^{\pi/3}$$

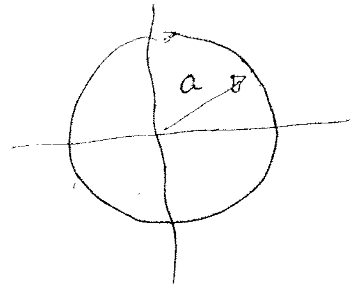
$$= \frac{1}{2}\left[2\sin 2\theta + 3\theta - 2\sin\theta\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left[\left(2\sin\frac{2\pi}{3} + 3\cdot\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right) - \left(2\sin\left(-\frac{2\pi}{3}\right) + 3\left(-\frac{\pi}{3}\right) - 2\sin\left(-\frac{\pi}{3}\right)\right)\right]$$

$$= 2\cdot\frac{\sqrt{3}}{2} + \pi - 2\cdot\frac{\sqrt{3}}{2}$$

$$= \pi$$

(27) $\iint_D \frac{1}{a^2+x^2+y^2}\, dx\,dy$



since $x^2+y^2 = r^2$

$$= \int_0^{2\pi}\int_0^a \frac{1}{a^2+r^2}\cdot r \; dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a \frac{1}{2}\cdot\frac{2r}{a^2+r^2}\, dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^a \frac{2r}{a^2+r^2}\, dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \cancel{\tfrac{1}{2}}\Big[\ln|a^2+r^2|\Big]_0^a d\theta$$

$$\cancel{= \frac{1}{2}\Big[\ln|a^2+x^2|\Big]_0^a}$$

$$= \frac{1}{2}\int_0^{2\pi} \ln|a^2+a^2|\; d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \ln|2a^2|\; d\theta$$

$$= \frac{1}{2}\left(\ln|2a^2|\right)\cdot\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\left(\ln|2a^2|\right)\cdot\big[\theta\big]_0^{2\pi}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\pi\cdot\ln|2a^2|$$

$$= \pi\ln|2a^2|$$